Chapter 8

Recurrence Relations and Generating Functions

And God said unto Moses:
"I am that I am."

– (Exodus, III, 14)

The concepts behind induction and recursion are intimately related. As we have seen in Chapter 3 – Mathematical Induction, many examples show that the properties of recursively defined functions or sets can be proved by mathematical induction. In the study of fundamental mathematics, "recursive" and "computable" carry the same meaning; that is, a function f can be recursively defined if and only if f can be computed by a program.

However, just being computable (being recursively definable) does not mean too much in the sense of efficiency. If it is possible, we want the function's "*closed-form*", so we can compute it efficiently. For example, we prefer to use $n(n+1)/2$ than a recursive program to compute the sum of the first n natural numbers.

A formula that recursively defines a function is called a "*recurrence relation*" or a "*recurrence equation*". Solving a recurrence equation means to find a close-form of the function defined by the recurrence equation. In this chapter, we emphasize on how to solve a given recurrence equation, few examples are given to illustrate why a recurrence equation solution of a given problem is preferable.

Some methods are suitable for solving certain kinds of recurrence equations, but there is no universal method to solve all kinds of recurrence equations. We have to, unfortunately, study different methods to find the solutions of different types of recurrence equations.

The generating function is an important subject in mathematics with application s in many diverse areas. Without too much pondering on the properties of the generation functions, we use it as a tool to solve some recurrence relations.

8.1 Recurrence Relations

A recurrence equation relates the value, a_n, of a sequence in terms of some or all of its past values, $a_{n-1}, a_{n-2}, \ldots$. In the most general form a recurrence equation is defined as follows:

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_0),$$

where f is a given function. Following are two illustrative examples of recurrence relations.

Example 8.1 $a_1 = 2$, and $a_n = 4a_{n-1} - 2$ for $n \geq 2$.

Example 8.2 $a_1 = 2, a_2 = 7$, and $a_n = a_{n-1}^2 + 2a_{n-2}$ for $n \geq 3$.

Example 8.3 $a_1 = 1, a_2 = 2, a_3 = 1$, and $a_n = 3a_{n-1} - a_{n-2} + 2a_{n-3}$ for $n \geq 4$.

Recurrence equations are also known as **difference equations**. Recurrence equations are valuable not only in mathematics and computer science but also in many other disciplines. In this chapter, our goal is to find an explicit solution of a recurrence equation and, more important, to understand the key mathematical ideas that lead to their solutions. Remember, sometimes it is natural to describe the solution of a problem in terms of a recurrence equations. The following example illustrates this point.

Example 8.4 A national car rental company allows customers one-way rental from one city to another city. Each month it finds that one fifth of the cars that start the month in New York City end it in Washington, D.C., and one sixth of the cars that start the month in Washington, D.C. end it in New York City. If the initial inventory in each city is 1000 cars, describe the situation after n months.

A solution of this problem is most conveniently obtained in terms of recurrence equations. If N_n and W_n denote the number of cars in the beginning of the nth month in New York City and Washington, D.C., respectively, then the number of cars in the beginning of the $n+1$th month satisfy:

$$\begin{aligned} N_{n+1} &= \frac{4}{5}N_n + \frac{1}{6}W_n \\ W_{n+1} &= \frac{1}{5}N_n + \frac{5}{6}W_n, \end{aligned}$$

where $N_0 = W_0 = 1000$. It is an easy observation that the values of N_n and W_n can be obtained for $n = 1, 2, \ldots$ from these equations. Another important exercise is to study the behavior of N_n and W_n as n gets larger. □

The following example shows that using recurrence relations sometimes can provide us an easier way to understand and solve some problems.

Example 8.5 Find the number of distinct partitions of a set of size n into k blocks. Let $A = \{a_1, a_2, \ldots, a_n\}$. A moments thought shows that the desired number is not easy to obtain; there are too many ways we can put n elements in k subsets. However, a recurrence equation is not difficult to build. Let $S_{n,k}$ denotes the desired number, i.e., the number of partitions of A into k blocks. Then,

$$S_{n+1,k+1} = S_{n,k} + (k+1)S_{n,k+1} \tag{8.1}$$

Why? A simple explanation follows. Let $B = A \cup \{a_{n+1}\}$ and we wish to obtain a partition of B into $(k+1)$ blocks. There are two possibilities.

Case 1: Consider a partition of A in $(k+1)$ blocks and put a_{n+1} in any one of the blocks. Since there are $S_{n,k+1}$ distinct partitions of A into $(k+1)$ blocks and a_{n+1} can be placed in any one of them, the total number of ways to do it is $(k+1)S_{n,k+1}$.

Case 2: Consider any partition of A in k blocks. and add to it the singleton set $\{a_{n+1}\}$. Thus, there are $S_{n,k}$ ways to achieve distinct partitions.

This justifies Equation (8.1). In addition, it is easy to verify that $S_{n,1} = S_{n,n} = 1$ for $n \geq 1$.

It may not be possible to find a closed form for $S_{n,k}$ but clearly, we can find the values of $S_{n,k}$ by using Equation (8.1). For example, $S_{1,1} = S_{2,1} = S_{2,2} = S_{3,1} = 1$ and

$$\begin{aligned} S_{3,k} &= S_{2,k} + 2S_{2,2} = 3, \\ S_{4,2} &= S_{3,1} + 2S_{3,2} = 1 + 2 \times 3 = 7, \\ S_{4,3} &= S_{3,2} + 3S_{3,3} = 3 + 3 = 6, \end{aligned}$$

etc. □

One systematic method for solving a recurrence equation is similar to the solution procedure of differential equations. Students familiar with the latter should not be surprised that the difference equations are discrete versions of differential equations.

In Example 8.1, $a_1 = 2$ is known as the initial condition and the other equation defines the recurrence. Similarly, in Example 8.2, $a_1 = 2, a_2 = 7$ are the initial conditions and the other equation defines the desired recurrence equations. It is possible to obtain all other values of the sequence by repeatedly substituting the previously obtained values. In Example 8.1, $a_2 = 4a_1 - 2 = 8 - 2 = 6, a_3 = 4a_2 - 2 = 24 - 2 = 22$, etc. However, one of the main purposes behind the study of recurrence equation is to be able to solve a given recurrence equation in closed form, i.e., to write the value of a_n in terms of n so that we don't have to evaluate it repeatedly by substitution. In this chapter we study how to solve linear recurrence equations, an important subclass of general recurrence equations.

8.1.1 Definitions

Definition 8.1: A recurrence equation for a_n is called **linear** if it can be written as a linear function of its past values, a_{n-1}, a_{n-2}, etc.

Example 8.1 above is a linear recurrence equations, whereas Example 8.2 is a nonlinear recurrence equation because a_n is a nonlinear function of a_{n-1}.

Definition 8.2: The **order** of a linear recurrence equation is k if a_n is a linear function of k past values $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$. In some cases, all of the past values may not be present on the left-hand side. For this reason the order of the recurrence equation is defined as the difference between the largest and smallest subscripts of the equation.

Example 8.1 above is a linear recurrence equation of order 1. Example 8.3 linear recurrence equation is of order 3.

Definition 8.3: A linear recurrence equation has **constant coefficients** if coefficients of a_{n-1}, a_{n-2}, etc., are all constant, i.e., do not depend on the indices n, $n-1$, etc.

For example, the recurrence equation in Example 8.3 has constant coefficients, whereas the recurrence equation $a_n = na_{n-1} + 4$ does not satisfy the constant coefficient criterion.

In this chapter we confine our attention to linear recurrence equation with constant coefficients of orders 1 and 2.

Definition 8.4: A linear recurrence equation is called **homogeneous** if a_n is a linear function of past values a_{n-1}, a_{n-2}, etc., only and does not contain any other additional terms; otherwise, it is called **nonhomogeneous**.

Example 8.1 above is a nonhomogeneous linear recurrence equation and the recurrence equation given in Example 8.3 is a homogeneous linear recurrence equation.

A nonhomogeneous linear recurrence equation consists of two parts—the nonhomogeneous component $f(n)$ and the rest of the equation, called the homogeneous component. In Example 8.1, the homogeneous part is $a_n = 4a_{n-1}$ and the nonhomogeneous component is $f(n) = -2$.

Definition 8.5: The solution of a given linear recurrence equation is called the **general** solution. It consists of two parts, the first part obtained from the homogeneous part and the second part contributed by the nonhomogeneous component $f(n)$. This general solution must satisfy the given set of initial conditions. A solution so obtained in know as **the particular solution**.

8.2 Solving Recurrence Relations

The solution of a given recurrence equation can be obtained by several methods; three of which are popular. When the recurrence equation is of order 1, it is convenient to obtain its solution by the method of **repeated substitutions**. The second method is similar to the solution procedure used to solve differential equations, and the third method is via **generating functions**. All of the methods have distinct advantages.

8.2.1 Repeated Substitution Method

This method is also called resubstitution method. As the name of this repeated substitution method indicates, the idea is to:

1. Substitute the values of a_{n-1} in the given equation, then the value of a_{n-2}, then the value of a_{n-3}, etc. These values are obtained from the given recurrence equation by replacing n by $n-1$, $n-2$, etc. in the defining equation.
2. Guess the solution of the recurrence equation from the above observations.
3. Prove the guessed result by mathematical induction.

Comment: The repeated substitution method works well for linear recurrence equations of order 1 and may or may not work for equations of order 2. Results can be obtained for the case when the coefficients in the linear recurrence equation are not constants.

Example 8.6 For $n \in \mathbf{N}^0$,

$$f(n) = \begin{cases} 2 & \text{if} \quad n = 0; \\ 3 + 2f(n-1) & \text{if} \quad n \geq 1. \end{cases}$$

$$\begin{aligned} f(n) &= 3 + 2f(n-1) \\ &= 3 + 2(3 + 2f(n-2)) \\ &= 3 \times (1+2) + 2^2 f(n-2) \\ &= 3 \times (1+2) + 2^2(3 + 2f(n-3)) \\ &= 3 \times (1+2+2^2) + 2^3 f(n-3) \\ &= \quad \cdots\cdots \\ &= 3(1+2+2^2+\cdots+2^{n-1}) + 2^n f(0) \\ &= 3 \times (2^n - 1) + 2^{n+1} \\ &= 5 \times 2^n - 3. \end{aligned}$$

Therefore, for $n \in \{0, 1, 2 \ldots\}$, $f(n) = 5 \times 2^n - 3$. □

Is this the correct solution? The verification can be made by mathematical induction, see the problem section.

Comment: One should compare the different representations for recurrence equations between Example 8.6 and the previous examples. The notations a_n and $f(n)$ carry the same meaning.

8.2.2 Characteristic Root Method

Solving Nonhomogeneous, Constant Coefficients, and Linear Difference Equations

This method is suitable to solve nonhomogeneous, constant coefficients and first or second-order linear difference equations. Theoretically, we can use this method to solve difference equations of order higher than 2, but higher order will introduce more characteristic roots and make the job more involved.

With this method the solution is obtained in several steps. First, we outline the major steps and then describe their implementation.

1. The first step is to remove the nonhomogeneous part from the given recurrence equation, thus obtaining the reduced homogeneous equation.
2. The second step is to find the solution of this homogeneous equation using the method described below. The solution so obtained contains some unknown coefficients that are determined later.
3. In the third step we obtain the solution associated with the nonhomogeneous part and combine it with the solution of the homogeneous part.
4. Finally, using the given initial conditions, and, if necessary, the first few values of the sequence obtained from the given equation, we obtain the unknown constants.

Solution of a Homogeneous Recurrence Equation

For convenience of presentation we consider a homogeneous recurrence equation with constant coefficients of order 2. Suppose that we wish to find a solution of

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

where c_1 and c_2 are two given constants. If $a_n = r^n$ is a solution of the given equation, then the following condition must be satisfied:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2}.$$

In other words, r is a solution of the quadratic equation $r^2 - c_1 r - c_2 = 0$. This quadratic equation is known as **the characteristic equation** of the given homogeneous recurrence equation. Suppose that r_1 and r_2 denote the two roots of the characteristic equation (for convenience of presentation we do not consider the case when the two roots are complex). It can then be verified that both $a_n = A_1 r_1^n$ and $a_n = A_2 r_2^n$ are solutions of the given homogeneous recurrence equation, where A_1 and A_2 are two constants. More generally, $a_n = A_1 r_1^n + A_2 r_2^n$ is a solution of the given homogeneous recurrence equation.

This general solution has only one remaining detail that requires further investigation. What will be the general solution of the given recurrence equation if $r_1 = r_2$? It would be unreasonable to suggest that the solution is given by $(A_1 + A_2) r_1^n$, because it is equivalent to a solution given by one root of the characteristic equation, whereas a quadratic equation has two roots. In the case of two equal roots the general solution of the given second-order recurrence equation is given by $(A_1 + A_2 n)r_1^n$.

In summary, to find the general solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Step 1: Obtain the associated characteristic equation $r^2 - c_1 r - c_2 = 0$.

Step 2: Find the two roots, r_1 and r_2, of this equation. If

1. the roots are different, then the general solution is $a_n = A_1 r_1^n + A_2 r_2^n$.
2. if the roots are equal, the the general solution is $a_n = (A_1 + A_2 n)r_1^n$.

Step 3: The two unknown constants, A_1 and A_2, are determined by the given initial conditions.

Solution of the Nonhomogeneous Part

In this subsection we consider a solution of the given recurrence equation that is governed by the nonhomogeneous part. First we consider the simple situations where the nonhomogeneous part, $f(n)$, is given as:

$$f(n) = a^n p(n),$$

where $p(n)$ is a polynomial in n of degree e. For example, $f(n) = 2^n(3n^2 + 4n + 5)$. Here $a = 2$ and $p(n) = (3n^2 + 4n + 5)$. Note that $p(n)$ is a polynomial in n of degree $e = 2$.

For such functions, the particular solution is governed by the characteristic equation $(r - a)^{e+1}$. Because this characteristic equation has $e + 1$ equal roots, its solution is $(A_1 n^e + A_2 n^{e-1} + \ldots + A_{e+1})a^n$. However, *is possible that a root of the characteristic equation of the homogeneous part may be equal to a*. For this reason, the characteristic equation $(r - a)^{e+1}$ is combined with the characteristic equation of the homogeneous part of the given recurrence equation equation and the new characteristic equation describes the general solution of the problem. We illustrate the complete procedure below.

We solve

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + a^n p(n),$$

where $p(n)$ is a polynomial in n of degree e.

The homogeneous part generates the characteristic equation $(r^2 - c_1 r - c_2)$ and the nonhomogeneous part generates the characteristic equation $(r - a)^{e+1}$. The combined characteristic equation is

$$(r^2 - c_1 r - c_2)(r - a)^{e+1} = (r - r_1)(r - r_2)(r - a)^{e+1},$$

where r_1 and r_2 are two roots of the quadratic equation $(r^2 - c_1 r - c_2) = 0$. Solution of the given recurrence equation depends on the values of r_1 and r_2. All possible cases are considered below. For convenience of presentation we take $e = 2$.

Case 1. If r_1, r_2, a are all distinct, then the general solution is

$$a_n = Ar_1^n + Br_2^n + (Cn^2 + Dn + E)a^n.$$

Case 2. If $r_1 = r_2, r_1 \neq a$, then the general solution is

$$a_n = (A + Bn)r_1^n + (Cn^2 + Dn + E)a^n.$$

Case 3. If $r_1 \neq r_2, r_2 = a$, then the general solution is

$$a_n = Ar_1^n + (Bn^3 + Cn^2 + Dn + E)a^n.$$

Similarly, if $r_1 \neq r_2, r_1 = a$, then the general solution is

$$a_n = Ar_2^n + (Bn^3 + Cn^2 + Dn + E)a^n.$$

Case 4. If $r_1 = r_2 = a$, then the general solution is

$$a_n = (An^4 + Bn^3 + Cn^2 + Dn + E)a^n.$$

Finally, we find the (five) unknown coefficients from the first five values of the sequence; two initial conditions are generally known, and three more values of the sequence can be generated from the given recurrence equation. Solution of a first-order nonhomogeneous recurrence equation can be obtained in exactly a similar manner; in this case the homogeneous component generates a characteristic equation of order 1.

In more general situations the nonhomogeneous component may contain more than one expressions. For example, suppose that the nonhomogeneous component of the given recurrence equation is:

$$f_1(n) + f_2(n) = a_1^n p_1(n) + a_2^n p_2(n).$$

In this case we find two characteristic equations—one for each $f_i(n)$ and combine it with the characteristic equation of the homogeneous part. Depending on the number of common roots, a polynomial of appropriate degree in n is generated. The following result is repeatedly applied to obtain the desired general solution from the combined characteristic equation.

> If a root, $r*$, of the characteristic equation has multiplicity m, then its contribution to the general solution of the given recurrence equation is a polynomial in n of degree $m - 1$ multiplied by $r*^n$, i.e., $(A_1 n^{m-1} + A_2 n^{m-2} + \ldots + A_{m-1} n + A_m) r*^n$.

We illustrate the above solution procedure by means of some examples.

Example 8.7 Solve the difference equation,

$$a_n = a_{n-1} + n^2, a_0 = 0.$$

Solution. This a first-order difference equation with nonhomogeneous component $f(n) = n^2$, which can be viewed as $n^2 \cdot 1^n$. Thus $p(n) = n^2$ is a polynomial in n of degree 2 and $a = 1$. The homogeneous part of the recurrence equation is $a_n - a_{n-1} = 0$ whose characteristic equation is $(r-1)$ and the nonhomogeneous component generates the characteristic equation $(r-1)^3$. Thus the combined characteristic equation is

$$(r-1)(r-1)^3 = (r-1)^4.$$

This characteristic equation one root, 1, with multiplicity 4. Hence the general solution of the given recurrence equation is

$$a_n = (An^2 + Bn^2 + Cn + D)1^n = An^3 + Bn^2 + Cn + D.$$

First four values of the sequence are $a_0 = 0,\ a_1 = 1,\ a_2 = 5,\ a_3 = 14$, one of them is the given initial condition and three more are obtained from the given recurrence equation.

Using these four values we find $A,\ B,\ C$, and D. We solve a system of four linear equations

$$\begin{aligned} D &= 0 \\ A + B + C + D &= 1 \\ 8A + 4B + 2C + D &= 5 \\ 27A + 9B + 3C + D &= 14 \end{aligned}$$

whose solution is

$$A = \frac{1}{3},\ A = \frac{1}{2},\ C = \frac{1}{6}, \text{ and } D = 0.$$

Therefore, the final answer of the given recurrence equation is

$$a_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

□

Example 8.8 Solve

$$a_n = 3a_{n-1} - 2a_{n-2} + 2^n + n^2,\ a_0 = 1,\ a_1 = 2.$$

Solution. This is a second-order, nonhomogeneous linear difference equation with $f(n) = 2^n + n^2$. It can be viewed as sum of two functions

$$f_1(n) = 2^n p_1(n), \quad \text{and } f_2(n) = 1^n p_2(n),$$

where $p_1(n) = 1$ is a polynomial of degree 0 and $p_2(n) = n^2$ is a polynomial of degree 2.

The homogeneous part of the equation has the characteristic equation

$$(r^2 - 3r + 2) = (r - 1)(r - 2)$$

and the nonhomogeneous parts have the characteristic equations $(r - 2)$ and $(r-1)^3$, respectively. Hence, the given recurrence equation's solution is obtained from the characteristic equation:

$$(r - 2)(r - 1)(r - 2)(r - 1)^3.$$

This equation has two distinct roots $r = 2$ and $r = 1$ of multiplicity 2 and 4 respectively. Hence, the general solution of the above characteristic equation is

$$a_n = (An + B)2^n + (Cn^3 + Dn^2 + En + F).$$

A, B, C, D, E, and F are obtained from the first two given conditions, $a_0 = 1$, $a_1 = 2$, and a_2, a_3, a_4, a_5 obtained from the given recurrence equation by substituting $n = 2, 3, 4, 5$ respectively.

We solve the system of six linear equations obtained by substituting $n = 0, 1, \ldots, 6$ into the given recurrence equation and obtain the final solution:

$$a_n = (2n + 8)2^n - (\frac{1}{3}n^3 + \frac{5}{2}n^2 + \frac{49}{6}n + 7).$$

8.2.3 Generating Function Method

In this section we consider basic properties of the generating functions and describe how to find the solution of a given difference equation.

Definition 8.6: Given a sequence of numbers $a_0, a_1, \ldots, a_n, \ldots$ the associated generating function is:

$$A(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots,$$

where z is a variable.

Let $A(z) = \sum_{n\geq 0} a_n z^n$, $B(z) = \sum_{n\geq 0} b_n z^n$, and $C(z) = \sum_{n\geq 0} c_n z^n$. We have the following properties:

1. $A(z) + B(z) = C(z)$ if and only if $\forall n \geq 0\ [a_n + b_n = c_n]$.
2. $A(z) \times B(z) = C(z)$ if and only if $\forall n \geq 0\ [c_n = \sum_{i=0}^{n} a_i \cdot b_{n-i}]$.

Some Useful Power Series and Their Closed Forms

In order to effectively use the generation functions, it is important to know some important sums. The following is a list of some useful power series and their closed forms. One may refer to the calculus textbooks for their proofs.

Let r be any real number and n be ant natural numbers.

$$\begin{aligned}
\frac{1}{1-sz} &= \sum_{0\le i} s^i z^i \\
\frac{1}{(1-sz)^2} &= \sum_{0\le i} (i+1) s^i z^i \\
(1+sz)^n &= \sum_{0\le i\le n} \binom{n}{i} s^i z^i \\
\frac{1}{(1+sz)^n} &= \sum_{0\le i} \binom{-n}{i} s^i z^i \\
\ln(1+sz) &= \sum_{1\le i} \frac{(-1)^{i+1}}{i} s^i z^i \\
-\ln(1-sz) &= \sum_{1\le i} \frac{1}{i} s^i z^i \\
e^{sz} &= \sum_{0\le i} \frac{1}{i!} s^i z^i
\end{aligned}$$

The connection between the closed forms of power series, generating functions, and recurrence relations can be seen in examples considered below.

Using Generating Functions to Solve Recurrence Equations

In brief, the key steps and idea behind the use of a generating function to solve a recurrence equation are:

Step 1: Write down the generating function of the given sequence.

Step 2: Use the given relation between a_n and a_{n-1} etc. to remove the recurrence from the generating function and simplify it.

Step 3: Expand the generating function obtained to in powers of z.

Step 4: Equate the coefficients of z^n in two different expressions of the generating functions of the same sequence. Thus obtain the solution of the given recurrence equation.

The following example explains the above steps.

Example 8.9 Consider the recurrence:

$$a_0 = 1, \text{ and for } n \geq 1, \ a_n = 2a_{n-1} + 1.$$

In the first step, all that we need to do is to write the generating function:

$$A(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots.$$

Next, by looking at the given recurrence relation we realize that for all values of n occurrences of $a_n - 2a_{n-1}$ can be replaced by 1. To utilize this property we multiply $A(z)$ by $2z$ and subtract it from $A(z)$, i.e., perform the following manipulations.

$$\begin{array}{rcccccccc}
A(z) = a_0 + & a_1 z + & a_2 z^2 + & a_3 z^3 & + & \cdots + & a_n z^n & + & \cdots \\
2zA(z) = \quad & + 2a_0 z + & 2a_1 z^2 + & 2a_3 z^3 & + & \cdots + & 2a_{n-1} z^n & + & \cdots \\
\hline
(1-2z)A(z) = a_0 + & 1z + & 1z^2 + & 1z^3 & + & \cdots + & 1z^n & + & \cdots
\end{array}$$

We have,

$$(1-2z)A(z) \ = \ 1 + z + z^2 + z^3 + \cdots \ = \ \frac{1}{1-z}$$

because $a_0 = 1$. Consequently,

$$A(z) \ = \ \frac{1}{(1-2z)(1-z)} \ = \ \frac{2}{1-2z} + \frac{-1}{1-z}.$$

We know that

$$\frac{2}{1-2z} = \sum_{0 \leq i} 2 \cdot 2^i z^i \ \text{ and } \ \frac{-1}{1-z} = \sum_{0 \leq i} (-1) \cdot z^i.$$

Thus,

$$A(z) \ = \ \sum_{0 \leq i} 2 \cdot 2^i z^i \ + \sum_{0 \leq i} (-1) \cdot z^i \ = \ \sum_{0 \leq i} (2^{i+1} - 1) z^i.$$

Therefore, $a_n = 2^{n+1} - 1$. □

8.2.4 An Example

In this subsection we solve a recurrence equation by all three methods discussed above. Consider the following recurrence equation:

$$a_0 = 1, \quad \text{and for all } n \geq 1, \quad 2a_n = a_{n-1} + 2^n.$$

Repeated Substitution Method

The recurrence equation can be rewritten as $a_n = \frac{1}{2}a_{n-1} + 2^{n-1}$ for **all** values of $n \geq 1$. Substituting $n-1$ in place of n gives $a_{n-1} = \frac{1}{2}a_{n-2} + 2^{n-2}$. Likewise, we can replace n by $n-2$. In the following development we make these substitutions and simplify:

$$\begin{aligned}
a_n &= \tfrac{1}{2}a_{n-1} + 2^{n-1} \\
&= \tfrac{1}{2}(\tfrac{1}{2}a_{n-2} + 2^{n-2}) + 2^{n-1} && (\text{substituting } a_{n-1} = \tfrac{1}{2}a_{n-2} + 2^{n-2}) \\
&= \tfrac{1}{2^2}a_{n-2} + 2^{n-3} + 2^{n-1} \\
&= \tfrac{1}{2^2}(\tfrac{1}{2}a_{n-3} + 2^{n-3}) + (2^{n-3} + 2^{n-1}) && (\text{substituting } a_{n-2} = \tfrac{1}{2}a_{n-3} + 2^{n-3}) \\
&= \tfrac{1}{2^3}a_{n-3} + 2^{n-5} + 2^{n-3} + 2^{n-1} \\
&\quad\vdots \\
&= \tfrac{1}{2^k}a_{n-k} + 2^{n-2k+1} + \cdots + 2^{n-3} + 2^{n-1} && (\text{guessing the answer}) \\
&\quad\vdots \\
&= \tfrac{1}{2^n}a_0 + 2^{n-2n+1} + \cdots + 2^{n-3} + 2^{n-1} && (\text{let } k = n) \\
&= \tfrac{1}{2^n} + 2^{-n+1} + \cdots + 2^{n-3} + 2^{n-1} && (a_0 = 1) \\
&= \tfrac{1}{2^n} + \tfrac{2^{-n+1} - 2^{n-1}2^2}{1-2^2} && (\text{sum of the geometric progression}) \\
&= \tfrac{1}{2^n} + \tfrac{1}{3}(2^{n+1} - 2^{-n+1}) \\
&= \tfrac{1}{2^n} + \tfrac{1}{3}(2 \cdot 2^n - 2 \cdot \tfrac{1}{2^n}) \\
&= \tfrac{1}{3} \cdot \tfrac{1}{2^n} + \tfrac{2}{3} \cdot 2^n.
\end{aligned}$$

Thus we have shown that the solution of the recurrence equation is $\frac{1}{3} \cdot \frac{1}{2^n} + \frac{2}{3} \cdot 2^n$. □

Characteristic Root Method

Since $a_n = \frac{1}{2}a_{n-1} + 2^{n-1}$, the characteristic polynomial is $(r - \frac{1}{2})(r-2)$, where $(r - \frac{1}{2})$ is contributed by the homogeneous part $a_n - \frac{1}{2}a_{n-1} = 0$ and $(r-2)$ is contributed by the nonhomogeneous part 2^n. Therefore, the general solution is

$$a_n = A(\frac{1}{2})^n + B2^n.$$

We know $a_0 = 1, a_1 = \frac{1}{2}a_0 + 2^0 = \frac{3}{2}$. To find the values of A and B we substitute $n = 0$ and $n = 1$ in the above general solution and solve the following equations:

$$\begin{array}{ccccccc}
1 & = & a_0 & = & A(\frac{1}{2})^0 + B2^0 & = & A + B \\
\frac{3}{2} & = & a_1 & = & A(\frac{1}{2})^1 + B2^1 & = & \frac{1}{2}A + 2B
\end{array}$$

We have $A = \frac{1}{3}$ $B = \frac{2}{3}$. Therefore, the solution is

$$a_n = \frac{1}{3} \cdot (\frac{1}{2})^n + \frac{2}{3} \cdot 2^n.$$

□

Generating Function Method

Consider the following generating function $G(z)$,

$$G(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{8.2}$$

Take $\frac{1}{2}z \times$ (8.2), we get

$$\frac{1}{2}zG(z) = \frac{1}{2}a_0 z + \frac{1}{2}a_1 z^2 + \frac{1}{2}a_2 z^3 + \cdots + \frac{1}{2}a_n z^{n+1} + \cdots \tag{8.3}$$

Take (8.2) − (8.3), and since $a_n - \frac{1}{2}a_{n-1} = 2^{n-1}$ we get

$$\begin{aligned}(1 - \frac{1}{2}z)G(z) =& a_0 + (a_1 - \frac{1}{2}a_0)z + (a_2 - \frac{1}{2}a_1)z^2 + \cdots + (a_n - \frac{1}{2}a_{n-1})z^n + \cdots \\ =& 1 + 2^{1-1}z + 2^{2-1}z^2 + \cdots + 2^{n-1}z^n + \cdots \\ =& 1 + \frac{1}{2}[2z + (2z)^2 + \cdots + (2z)^n + \cdots] \\ =& 1 + \frac{1}{2}\frac{2z}{1-2z} \qquad \text{(sum of the geometric progression)} \\ =& 1 + \frac{z}{1-2z}.\end{aligned}$$

Therefore,

$$\begin{aligned} G(z) &= \frac{1}{1-\frac{1}{2}z} + \frac{1}{(1-\frac{1}{2}z)(1-2z)} \\ &= \frac{1}{1-\frac{1}{2}z} + \frac{2z}{(2-z)(1-2z)}. \end{aligned} \tag{8.4}$$

Now, we want to decompose the second term in the right hand side of (8.4) in the following form:

$$\begin{aligned} \frac{2z}{(2-z)(1-2z)} &= \frac{A}{2-z} + \frac{B}{1-2z} \\ &= \frac{A - 2zA + 2B - Bz}{(2-z)(1-2z)} \\ &= \frac{(A+2B) + (-2A-B)z}{(2-z)(1-2z)}. \end{aligned}$$

We have $(A + 2B) = 0$ and $(2A + B) = -2$, and solve the equations to have $A = -\frac{4}{3}$ and $B = \frac{2}{3}$. Therefore, the generating function $G(z)$ is

$$\begin{aligned} G(z) &= \frac{1}{1-\frac{1}{2}z} + \frac{-\frac{4}{3}}{2-z} + \frac{\frac{2}{3}}{1-2z} \\ &= \frac{1}{1-\frac{1}{2}z} - \frac{2}{3} \cdot \frac{1}{1-\frac{1}{2}z} + \frac{2}{3} \cdot \frac{1}{1-2z} \\ &= \frac{1}{3} \cdot \frac{1}{1-\frac{1}{2}z} + \frac{2}{3} \cdot \frac{1}{1-2z}. \end{aligned}$$

Therefore, the solution is

$$a_n = \frac{1}{3} \cdot (\frac{1}{2})^n + \frac{2}{3} \cdot 2^n.$$

□

Notation: $\mathbf{N}^0 = \{0, 1, 2, 3, \ldots\}$, $\mathbf{N} = \{1, 2, 3, \ldots\}$, and $\mathbf{R}$ the set of real numbers.

8.3 Problems

In Problems 1 through 6 find a recurrence relation in terms of previous values. Also give the boundary conditions.

Problem 1: Find a recurrence relation for the sum of the first n positive odd integers in terms of sum of the first $(n-1)$ positive odd integers.

Problem 2: Find a recurrence relation for the maximum number of pieces of a pizza made by n straight cuts.

Problem 3: Find a recurrence relation for the number of ways to put n cents in a machine using identical pennies, nickels, dimes, and quarters.

Problem 4: Recall the recurrence relation for r-combinations from n objects: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. Find a similar recurrence relation for r-permutations from n objects; i.e., for $P(n, r)$.

Problem 5: Find a recurrence relation for the number of n digit binary sequences with no consecutive 1's.

Problem 6: Find a recurrence relation for the maximum number of nodes in a binary tree of depth d.

Problem 7: Find a recurrence relation for the number of ways to fully parenthesize an expression of n variables:

$$x_1 + x_2 + x_3 + \cdots + x_n.$$

For example, $((x_1 + x_2) + x_3), (x_1 + (x_2 + x_3))$ are the only two ways to fully parenthesize $x_1 + x_2 + x_3$. Parenthesization is an important subject in compilers.

Hint: Use the following as a starting point. For $i = 1, \ldots, n$,

$$x_1 + \cdots + x_n = ((x_1 + \cdots + x_i) + (x_{i+1} + \cdots + x_n)).$$

Use the Repeated Substitution Method to guess the solutions of the recurrence relations in Problems 8 through 11 and verify the correctness of your guesses by mathematical induction.

Problem 8: Solve the following recurrence relation: For $n \in \mathbf{N}^0$,

$$f(n) = \begin{cases} 2 & \text{if } n = 0; \\ 3 + f(n-1) & \text{if } n \geq 1. \end{cases}$$

Problem 9: Solve the following recurrence relation: For $n \in \mathbf{N}^0$,

$$f(n) = \begin{cases} 2 & \text{if} \quad n = 0; \\ 3f(n-1) + 2 & \text{if} \quad n \geq 1. \end{cases}$$

Problem 10: Solve the following recurrence relation: For $n \in \mathbf{N}$,

$$f(n) = \begin{cases} 1 & \text{if} \quad n = 1; \\ 3f(\lceil \frac{n}{3} \rceil) & \text{if} \quad n \geq 2. \end{cases}$$

Problem 11: Solve the following recurrence relation: For $n \in \mathbf{N}$,

$$f(n) = \begin{cases} c & \text{if} \quad n = 1; \\ af(\lceil \frac{n}{b} \rceil) + cn & \text{if} \quad n \geq 2, \end{cases}$$

where $b \in \mathbf{N}, a, c \in \mathbf{R}$, and $b > 1$.

Problem 12: Use the characteristic root method to solve the following linear homogeneous recurrence relation.

Let $f(0) = 1, f(1) = -1$, and for $n \in \mathbf{N}^0$,

$$f(n+2) + 2f(n+1) - 3f(n) = 0.$$

Problem 13: Let f_n be the Fibonacci numbers, i.e., $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Define a_n as

$$a_n = \frac{f_n}{f_{n-1}}, \qquad \text{for } n \in \mathbf{N}.$$

Give a recurrence relation to compute a_n and solve the relation.

Hint: Assume that a_n converges to r as $n \to \infty$.

Problem 14: Solve the following recurrence relation:

$$\begin{cases} a_0 = 0, \\ a_1 = -1, \\ a_n - 7a_{n-1} + 12a_{n-2} = 0 \quad \text{for } n \geq 2. \end{cases}$$

Problem 15: Solve the following recurrence relation:

$$\begin{cases} a_1 = 1, \\ a_2 = 1, \\ a_n + 2a_{n-1} - 15a_{n-2} = 0 \quad \text{for } n \geq 3. \end{cases}$$

Problem 16: Solve the following recurrence relation:

$$\begin{cases} a_0 = 2, \\ a_1 = 0, \\ -2a_n + 18a_{n-2} = 0 \quad \text{for } n \geq 2. \end{cases}$$

Problem 17: Solve the following recurrence relation:

$$\begin{cases} a_1 = 2, \\ a_2 = 6, \\ a_n - 4a_{n-1} + 4a_{n-2} = 0 \quad \text{for } n \geq 3. \end{cases}$$

Problem 18: Solve the following recurrence relation:

$$\begin{cases} a_1 = 5, \\ a_2 = -5, \\ a_n + 6a_{n-1} + 9a_{n-2} = 0 \quad \text{for } n \geq 3. \end{cases}$$

Problem 19: Solve the following recurrence relation:

$$\begin{cases} a_0 = 1, \\ a_1 = 2, \\ a_n - 5a_{n-1} + 6a_{n-2} = 2n + 1 \quad \text{for } n \geq 2. \end{cases}$$

Problem 20: Solve the following recurrence relation:

$$\begin{cases} a_0 = 1, \\ a_1 = -1, \\ a_n - 3a_{n-1} + 2a_{n-2} = n \quad \text{for } n \geq 2. \end{cases}$$

Problem 21: Given $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, $a_3 = 13$, and

$$a_n + ba_{n-1} + ca_{n-2} = 0 \quad \text{for } n \geq 2.$$

where b and c are two unknown numbers. Find a_n in a closed form.

Problem 22: Given $a_0 = 0$, $a_1 = 1$, and

$$3a_n - 10a_{n-1} + 3a_{n-2} = 3^n \quad \text{for } n \geq 2,$$

Solve the equation.

Problem 23: Given $a_0 = 0$, $a_1 = 1$, and

$$5a_n - 6a_{n-1} + a_{n-2} = n^2(\frac{1}{5})^n \quad \text{for } n \geq 2.$$

Solve the equation.

8.4 Solutions

Solution 1: If $n = 1$, the sum of the first n positive odd integers is simply 1. If $n \geq 2$, the sum of the first n positive odd integers is the n^{th} positive odd integer plus the sum of the first $n-1$ positive odd integers. Therefore, we can define the sum of the first n positive odd integers, $f(n)$, in the following difference equation. For $n \in \mathbf{N}$,

$$f(n) = \begin{cases} 1 & \text{if } n = 1; \\ f(n-1) + (2n-1) & \text{if } n \geq 2. \end{cases}$$

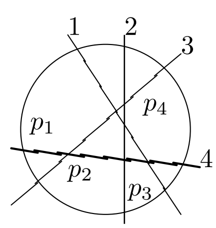
Note: the n^{th} positive odd integer is $2n-1$ not $2n+1$.

Alternatively, you can let n range over $\mathbf{N}^0$, and let $f(0) = 0$ as the initial point of the difference equation. □

Solution 2: Let $f(n)$ be the maximum number of pieces into which a pizza is cut by n straight cuts.

It's clear that if $n = 0$, we have the whole pizza. Thus, $f(0) = 1$.

To maximize the number of pieces of the pizza by n straight cuts we must not use parallel cuts and we must not have any three cuts which intersect at the same point. In other words, if we have a pizza with $n-1$ cuts and obtain the maximum number of pieces, then the n^{th} cut must cross all previous $n-1$ cuts at new $n-1$ different points. That means, the n^{th} cut must run through n of the $f(n-1)$ pieces by the $n-1$ previous cuts. Due to the n^{th} cut, each of the n pieces will be cut into two pieces. Therefore, the n^{th} cut will introduce n more pieces. For example, in the following figure the 4^{th} cut has cut each of the pieces p_1, p_2, p_3 and p_4 into two parts.



We can use the following difference equation to represent the numbers:

Let $n \in \mathbf{N}^0$.

$$f(n) = \begin{cases} 1 & \text{if } n = 0; \\ f(n-1) + n & \text{if } n \geq 1. \end{cases}$$

□

Solution 3: Let $n \in \mathbf{N}^0$, and $f(n)$ be the number of ways to put n cents into a machine by using identical pennies, nickels, dimes, or quarters.

For $n = 0$, the only way is: do nothing, so we have $f(0) = 1$. For $1 \le n \le 4$, we have to put in all pennies. Thus, $f(n) = 1$ for $1 \le n \le 4$.

When $n = 5$, let's consider the first coin we will put into the machine. We can put in a penny or a nickel. If we put in a penny, then we have $f(5-1)$ ways to put in the rest of the 4 cents. If we put in a nickel, then we have $f(5-5)$ ways to put in the rest of the 0 cents. Therefore, we have $f(5) = f(5-1) + f(5-5)$ ways to put in 5 cents, i.e., $f(5) = 1 + 1 = 2$. The two ways are

$$1[1111] \text{ and } 5[\,],$$

where [1111] is the way to put in 4 cents, and [] is the way to put in 0 cents. For $6 \le n \le 9$, the situation is similar to the case $n = 5$: we have two choices for the 1st coin (a penny and a nickel). Thus, we have

$$f(n) = f(n-1) + f(n-5) \quad \text{for } 5 \le n \le 9.$$

For example, if $n = 6$, we have the following ways:

$$1[11111],\ 1[5],\ \text{ and } 5[1],$$

where [11111] and [5] are the ways to put in $6 - 1$ cents, and [1] is the way to put in $6 - 5$ cents. The idea is the same for larger values of n. For $10 \le n < 25$, we have 3 choices for the 1st coin (a dim is another option). Likewise, when $n \ge 25$, we can use any kind of the coins for the 1st coin. Thus, we have the following difference equation describes all 4 possible cases. For $n \in \mathbf{N}^0$,

$$f(n) = \begin{cases} 1 & \text{if } 0 \le n \le 4; \\ f(n-1) + f(n-5) & \text{if } 5 \le n \le 9; \\ f(n-1) + f(n-5) + f(n-10) & \text{if } 10 \le n \le 24; \\ f(n-1) + f(n-5) + f(n-10) + f(n-25) & \text{if } 25 \le n. \end{cases}$$

□

Solution 4: Let $n, r \in \mathbf{N}^0$ and $P(n, r)$ denote the number of r-permutations from n distinct objects.

It is clear that $P(n,0) = 1$ and $P(n,r) = 0$ for $n < r$. Suppose $r \neq 0$ and $n \geq r$. Consider the selection procedure with respect to n^{th} object. We have the following two choices.

1. The n^{th} object is not selected. In this case, we permute r objects out of the remaining $n-1$ objects in $P(n-1,r)$ ways.
2. The n^{th} object is selected. In this case, we first find $(r-1)$-permutations from the remaining $n-1$ objects in $P(n-1,r-1)$ ways. For each $(r-1)$-permutation, we have to insert the n^{th} object into it, and we have r way to do so. In another words, we can have r different r-permutations from each $(r-1)$-permutation. Therefore, we have $rP(n-1,r-1)$ r-permutations.

By the rule of sum, we have

$$P(n,r) = P(n-1,r) + rP(n-1,r-1).$$

□

Solution 5: Let $n \in \mathbf{N}$, and $f(n)$ be the number of n digit binary sequences with no consecutive 1's. Some of the initial few sequences are shown below:

$n=1$	$n=2$	$n=3$	$n=4$
0	0 0	00 0	000 0
1	1 0	10 0	100 0
	01	01 0	010 0
		0 01	001 0
		1 01	101 0
			00 01
			10 01
			01 01

From the table, $f(1) = 2$ and $f(2) = 3$. For $n = 3$, we obtain the valid strings as follows: (i) We take all valid strings of length 2 and at the end of each we attach 0. (ii) We take all valid strings of length 1 and at the end of each we attach 01. Likewise, for $n = 4$, the valid strings are all valid strings of length 3 attached with 0 and all valid strings of length 2 attached with 01. In general, we have the following procedures in two cases to generate valid strings of length $n \geq 3$ based on the previous valid strings.

i. Take the valid strings of length $n-1$ and at the end of each string attach 0. We obtain$f(n-1)$ new strings in this case.

ii. Take the valid strings of length $n-2$ and at the end of each string attach 01. We obtain $f(n-2)$ new strings in this case.

It is clear that both procedures above generate valid strings of length 2. But we have to argue that the string generated by the two procedures cover all valid strings of length n. Let s be a valid string of length $n \geq 3$ and let s' and s'' denote two substrings of s, where s' consists of the first $n-1$ digits of s and s'' consists of the first $n-2$ digits of s. There are two cases:

1. The n^{th} digit of s is 0. In this case, s' can be any valid string of length $n-1$. Thus, the procedure described in i. generate s.
2. The n^{th} digit of s is 1. In this case the $(n-1)^th$ digit of s must be 0 and the s'' can be any valid string of length $n-2$. Thus, the procedure described in ii. generate s.

Finally, by the rule of sum, we have

$$f(n) = \begin{cases} 2 & \text{if } n = 1; \\ 3 & \text{if } n = 2; \\ f(n-1) + f(n-2) & \text{if } n \geq 3. \end{cases}$$

□

Solution 6: To maximize the number of nodes in a binary tree, we have to maximize the number of nodes at each level[1]. Let's first find a recurrence relation, $l(d)$, and its solution for the maximum number of nodes at level d in a binary tree. It's clear that $l(0) = 1$, and $l(d) = 2 \times l(d-1)$ for $d \geq 2$, because the root is the only node at level 0, and each node at level $d-1$ must have two children in order to maximize the number of nodes at the next level. Therefore,

$$l(d) = 2l(d-1) = 2^2 l(d-2) = \cdots = 2^d.$$

Let $n(d)$ be the maximum number of nodes in a binary tree of depth d. We note that $n(0) = 1$, and for $d \geq 1$, $n(d)$ is the the maximum number of nodes in a binary tree of depth $d-1$ plus the the maximum number of nodes at level d. Thus, we obtain the difference equation:

$$n(d) = \begin{cases} 1 & \text{if } d = 0; \\ n(d-1) + 2^d & \text{if } d \geq 1. \end{cases}$$

□

[1]We define that the root is at level 0, and any child of a node at level k is at level $k+1$. Thus, the leaves of a binary tree of depth d are at level $\leq d$.

Solution 7: Let $f(n)$ be the number of ways to fully parenthesize the expression:

$$x_1 + x_2 + x_3 + \cdots + x_n.$$

When $n = 1$, we have (x_1) only, thus $f(1) = 1$. (In general we omit the parentheses when $n = 1$.) When $n = 2$, (x_1+x_2) is the only way to parenthesize. Thus $f(2) = 1$. When $n = 3$, we have two ways to parenthesize:$(x_1 + (x_2 + x_3))$ and $((x_1 + x_2) + x_3)$. Therefore,$f(3) = 2$.

In general, after we put the outermost pair of parentheses, we can exhaustively list all possible ways to put the second parenthesize the second outer pairs of parentheses.

$$\begin{array}{r|l}
\text{case} & \\
1 & (x_1 + (x_2 + x_3 \ldots + x_n)); \\
2 & ((x_1 + x_2) + (x_3 \ldots + x_n)); \\
. & \quad \cdots\cdots\cdots \\
i & ((x_1 + \cdots + x_i) + (x_{i+1} + \cdots + x_n)); \\
. & \quad \cdots\cdots\cdots \\
n-1 & ((x_1 + \ldots + x_{n-1}) + x_n).
\end{array}$$

Note that $((\cdots) + (\cdots) + (\cdots))$ is not fully parenthesized because it can be further parenthesized as $(((\cdots) + (\cdots)) + (\cdots))$ or $((\cdots) + ((\cdots) + (\cdots)))$. Consider the i^{th} case above. We have $f(i)$ ways to parenthesize the first part, $x_1 + \cdots + x_i$, and $f(n-i)$ ways to parenthesize the second part, $x_{i+1} + \cdots + x_n$. By the product rule, there are $f(i) \times f(n-i)$ ways to further parenthesize in the case. For $i = 1, \cdots, n-1$, by the sum rule, we obtain the following equation:

$$f(n) = \begin{cases} 1 & \text{if } n = 1; \\ \displaystyle\sum_{i=1}^{n-1} f(i)f(n-i) & \text{if } n \geq 2. \end{cases}$$

□

Solution 8: To solve the recurrence equation:

$$f(n) = \begin{cases} 2 & \text{if} \quad n = 0; \\ 3 + f(n-1) & \text{if} \quad n \geq 1, \end{cases}$$

we apply the resubstitution method. Thus,

$$\begin{aligned} f(n) &= 3 + f(n-1) \\ &= 3 + (3 + f(n-2)) \\ &= 3 \times 2 + f(n-2) \\ &= 3 \times 2 + (3 + f(n-3)) \\ &= 3 \times 3 + f(n-3) \\ &= \quad \cdots\cdots \\ &= 3n + f(0) \\ &= 3n + 2. \end{aligned}$$

Therefore, for $n \in \mathbf{N}^0$, $f(n) = 3n + 2$. □

Prove by Mathematical Inductions:

The basis step is obviously satisfied: $f(0) = 0 + 2 = 2$. For hypothesis, assume that $f(n) = 3n+2$. For inductive step, we use the definition for $f(n+1)$ and the hypothesis to have

$$\begin{aligned} f(n+1) &= 3 + f(n) \\ &= 3 + (3n + 2) \\ &= 3(n+1) + 2. \end{aligned}$$

□

Solution 9: Given

$$f(n) = \begin{cases} 2 & \text{if} \quad n = 0; \\ 3f(n-1) + 2 & \text{if} \quad n \geq 1, \end{cases}$$

By applying the resubstitution method, we obtain:

$$\begin{aligned} f(n) &= 3f(n-1) + 2 \\ &= 3(3f(n-2) + 2) + 2 \\ &= 3^2 f(n-2) + 3 \cdot 2 + 2 \\ &= 3^2(3f(n-3) + 2) + 3 \cdot 2 + 2 \\ &= 3^3 f(n-3) + 3^2 \cdot 2 + 3 \cdot 2 + 2 \\ &= \quad \cdots\cdots \\ &= 3^n f(0) + 3^{n-1} \cdot 2 + \cdots + 3 \cdot 2 + 2 \\ &= 3^n \cdot 2 + 3^{n-1} \cdot 2 + \cdots + 3 \cdot 2 + 2 \\ &= 3^{n+1} - 1. \end{aligned}$$

Therefore, $f(n) = 3^{n+1} - 1$ for $n \in \mathbf{N}^0$.

Prove by Mathematical Inductions:

The basis step is satisfied because $f(0) = 3^1 - 1 = 2$. For hypothesis, assume that $f(n) = 3^{n+1} - 1$. For inductive step, we use the definition for $f(n+1)$ and the hypothesis to have

$$\begin{aligned} f(n+1) &= 3f(n) + 2 \\ &= 3(\cdot 3^{n+1} - 1) + 2 \\ &= 3^{n+2} - 1. \end{aligned}$$

□

Solution 10: First, we prove that if $n, a, b \in \mathbf{N}$, then

$$\left\lceil \frac{\lceil \frac{n}{a} \rceil}{b} \right\rceil = \left\lceil \frac{n}{ab} \right\rceil . \tag{8.5}$$

Let $n = abp + r$, where $0 \le r < ab$, and p is an integer. If $r = 0$, the equality (8.5) follows immediately. If $r \ne 0$, i.e., $0 < r < ab$, then

$$\left\lceil \frac{n}{ab} \right\rceil = \left\lceil \frac{abp + r}{ab} \right\rceil = p + 1, \quad \text{and}$$

$$\left\lceil \frac{n}{a} \right\rceil = \left\lceil \frac{abp + r}{a} \right\rceil = \left\lceil bp + \frac{r}{a} \right\rceil = bp + k, \quad 1 \le k \le b,$$

Becuse $(0 < r < ab)$ implies $(0 < \frac{r}{a} < b)$, which, in turn, implies that the smallest integer containing $\frac{r}{a}$ cannot bigger than b. And, since $(1 \le k \le b)$ implies $(\frac{1}{b} \le \frac{k}{b} \le 1)$, we have

$$\left\lceil \frac{\lceil \frac{n}{a} \rceil}{b} \right\rceil = \left\lceil \frac{bp + k}{b} \right\rceil = \left\lceil p + \frac{k}{b} \right\rceil = p + 1 = \left\lceil \frac{n}{ab} \right\rceil .$$

Therefore,

$$\left\lceil \frac{\lceil \frac{n}{3^k} \rceil}{3} \right\rceil = \left\lceil \frac{n}{3^{k+1}} \right\rceil, \quad k \ge 0. \tag{8.6}$$

□

To solve the recurrence equation:

$$f(n) = \begin{cases} 1 & \text{if} \quad n = 1; \\ 3f(\lceil \frac{n}{3} \rceil) & \text{if} \quad n \ge 2, \end{cases}$$

we use the result given in (8.6) and the resubstitution method; we have

$$
\begin{aligned}
f(n) &= 3f(\left\lceil \frac{n}{3} \right\rceil) \\
&= 3(3f(\left\lceil \frac{\lceil \frac{n}{3} \rceil}{3} \right\rceil)) \\
&= 3^2 f(\left\lceil \frac{n}{3^2} \right\rceil) \\
&= 3^3 f(\left\lceil \frac{n}{3^3} \right\rceil) \\
&= \quad \cdots \\
&= 3^k f(\left\lceil \frac{n}{3^k} \right\rceil) \quad \text{where } k = \lceil \log_3^n \rceil \\
&= 3^k f(1) \\
&= 3^{\lceil \log_3^n \rceil}.
\end{aligned}
$$

Note: If n is a power of 3, then $f(n) = n$. □

Prove by Mathematical Inductions:

The basis step is satisfied because $f(1) = 3^{\lceil \log_3^1 \rceil} = 3^0 = 1$. For hypothesis, assume that $f(n) = 3^{\lceil \log_3^n \rceil}$. For inductive step, we discuss in two cases of n.

case 1: $n = 3^k$ where $k \geq 1$.

$$
\begin{aligned}
f(n+1) &= f(3^k + 1) \\
&= 3f(\left\lceil \frac{3^k + 1}{3} \right\rceil) \\
&= 3f(\left\lceil 3^{k-1} + \frac{1}{3} \right\rceil) \\
&= 3f(3^{k-1} + 1) \\
&= 3 \cdot 3^{\left(\left\lceil \log_3^{(3^{k-1}+1)} \right\rceil\right)}.
\end{aligned} \tag{8.7}
$$

Since $3^{k-1} < 3^{k-1} + 1 < 3^k$, thus (8.7) $= 3 \cdot 3^k = 3^{k+1}$. And, since $3^k < 3^k + 1 < 3^{k+1}$, thus $\left\lceil \log_3^{3^k+1} \right\rceil = k + 1$. Therefore, $f(n+1) = 3^{\lceil \log_3^{n+1} \rceil}$.

case 2: $3^{k-1} < n < 3^k$ where $k \geq 1$.

$$\begin{aligned} 3^{k-1} < n < 3^k & \Rightarrow 3^{k-1}+1 < n+1 < 3^k+1 \\ & \Rightarrow 3^{k-1}+1 < n+1 \leq 3^k \qquad (8.8) \\ & \Rightarrow 3^{k-2}+\frac{1}{3} < \frac{n+1}{3} \leq 3^{k-1} \\ & \Rightarrow k-2 < \log_3^{\lceil \frac{n+1}{3} \rceil} \leq k-1 \\ & \Rightarrow \left\lceil \log_3^{\lceil \frac{n+1}{3} \rceil} \right\rceil = k-1. \end{aligned}$$

$$\begin{aligned} f(n+1) & = 3f(\left\lceil \frac{n+1}{3} \right\rceil) \\ & = 3 \cdot 3^{\left\lceil \log_3^{\lceil \frac{n+1}{3} \rceil} \right\rceil} \\ & = 3 \cdot 3^{k-1} \\ & = 3^k. \end{aligned}$$

From (8.8), $k-1 < \log_3^{(n+1)} \leq k$. Thus, $\left\lceil \log_3^{(n+1)} \right\rceil = k$. Therefore $f(n+1) = 3^{\left\lceil \log_3^{(n+1)} \right\rceil}$. □

Solution 11: We also repeatedly use the result in (8.6) and the resubstitution method to solve the recurrence equation:

$$f(n) = \begin{cases} c & \text{if} \quad n = 1; \\ af(\lceil \frac{n}{b} \rceil) + cn & \text{if} \quad n \geq 2, \end{cases}$$

where $a, c \in \mathbf{R}$, $b \in \mathbf{N}$ and $b > 1$.

Resubstitution Method:

$$\begin{aligned} f(n) & = af(\lceil \tfrac{n}{b} \rceil) + cn \\ & = a\left(af(\lceil \tfrac{\lceil \frac{n}{b} \rceil}{b} \rceil) + c\lceil \tfrac{n}{b} \rceil\right) + cn \\ & = a^2 f(\lceil \tfrac{n}{b^2} \rceil) + ac\lceil \tfrac{n}{b} \rceil + cn \\ & = a^3 f(\lceil \tfrac{n}{b^3} \rceil) + a^2 c\lceil \tfrac{n}{b^2} \rceil + ac\lceil \tfrac{n}{b} \rceil + cn \\ & = \cdots\cdots \\ & = a^k f(\lceil \tfrac{n}{3^k} \rceil) + a^{k-1}c\lceil \tfrac{n}{b^{k-1}} \rceil + \cdots + a^2 c\lceil \tfrac{n}{b^2} \rceil + ac\lceil \tfrac{n}{b} \rceil + cn \ \ k = \lceil \log_b^n \rceil \\ & = a^k c + a^{k-1}c\lceil \tfrac{n}{b^{k-1}} \rceil + \cdots + a^2 c\lceil \tfrac{n}{b^2} \rceil + ac\lceil \tfrac{n}{b} \rceil + cn \\ & = a^k c\lceil \tfrac{n}{b^k} \rceil + a^{k-1}c\lceil \tfrac{n}{b^{k-1}} \rceil + \cdots + a^2 c\lceil \tfrac{n}{b^2} \rceil + ac\lceil \tfrac{n}{b} \rceil + a^0 c\lceil \tfrac{n}{b^0} \rceil \\ & = c\sum_{i=0}^{\lceil \log_b^n \rceil} a^i \lceil \tfrac{n}{b^i} \rceil. \end{aligned}$$

It is difficult to further simplify the above formula. But in many applications, e.g., the algorithm analysis, an asymptotic solution is accurate enough for our purpose, e.g., O gives a upper bound, Ω gives a lower bound, and Θ gives an approximation. In order to get rid of the ceiling functions in the formula we assume that n is a power of b or simply. In such a way, we can further simplify the formula in the following.

$$\begin{aligned} c\sum_{i=0}^{\log_b^n} a^i \frac{n}{b^i} &= cn\sum_{i=0}^{\log_b^n} \left(\frac{a}{b}\right)^i \\ &= cn\frac{1-\left(\frac{a}{b}\right)^{(\log_b^n+1)}}{1-\frac{a}{b}} \qquad \text{assume } a \neq b \\ &= cn\frac{b}{a-b}\left(\frac{a^{\log_b^n+1}}{nb}-1\right) \\ &= \frac{ca^{\log_b^n+1}}{a-b} - \frac{cnb}{a-b}. \end{aligned} \tag{8.9}$$

Thus, we have the following results.

$$(8.9) \in \begin{cases} \Theta(n) & \text{if } a < b; \\ \Theta(n\log n) & \text{if } a = b; \\ \Theta(n^{\log_b^a}) & \text{if } a > b. \end{cases}$$

Note 1: A precise mathematical inductive proof of the result is very involved due to the ceiling function. You can assume that n ranges over the power numbers of b.

Note 2: For this problem, b must be a natural number because, otherwise, $\lceil\frac{\lceil\frac{n}{b}\rceil}{b}\rceil = \lceil\frac{n}{b^2}\rceil$ is incorrect in general. Here is a counter example: if $n = 2$ and $b = 1.5$, then $\lceil\frac{n}{b^2}\rceil = 1$ and $\lceil\frac{\lceil\frac{n}{b}\rceil}{b}\rceil = 2$.

□

Solution 12: Let $f(0) = 1, f(1) = -1$, and for $n \geq 2$,

$$f(n+2) + 2f(n+1) - 3f(n) = 0.$$

We solve the equation by the characteristic root method. We first find characteristic polynomial, $r^2 + 2r - 3$, and solve its associated characteristic equation, $r^2 + 2r - 3 =$, to obtain its two roots: $r =$ and $r = -31$. Since the two roots are distinct, the general solution to the equation is:

$$f(n) = A \cdot 1^n + B(-3)^n.$$

The initial conditions $f(0) = 1$ and $f(1) = -1$ generate the following equations:

$$\begin{aligned} 1 &= A \cdot 1^0 + B \cdot (-3)^0 \\ -1 &= A \cdot 1^1 + B \cdot (-3)^1 \end{aligned}$$

Solve the above equations to get $A = B = \frac{1}{2}$. That gives the solution:

$$\begin{aligned} f(n) &= \frac{1}{2} \cdot 1^n + \frac{1}{2} \cdot (-3)^n \\ &= \frac{1}{2} + \frac{1}{2} \cdot (-3)^n. \end{aligned}$$

□

Solution 13: Let f_n be the Fibonacci numbers, i.e., $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. Define

$$a_n = \frac{f_n}{f_{n-1}}, \qquad \text{for } n \in \mathbf{N}.$$

For $n = 1$, $a_1 = \frac{f_1}{f_0} = \frac{1}{1} = 1$, and for $n \geq 2$,

$$a_n = \frac{f_n}{f_{n-1}} = \frac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \frac{f_{n-2}}{f_{n-1}} = 1 + \frac{1}{f_{n-1}/f_{n-2}} = 1 + \frac{1}{a_{n-1}}.$$

Therefore,

$$a_n = \begin{cases} 1 & \text{if } n = 1; \\ 1 + \frac{1}{a_{n-1}} & \text{if } n \geq 2. \end{cases}$$

□

We are interested in the value of $\lim_{n\to\infty} a_n$. Assume $\lim_{n\to\infty} a_n = r$. That means, when n approximates to infinity, $a_n = a_{n-1} = r$. Thus, we have

$$\begin{aligned} \lim_{n\to\infty} a_n = 1 + \frac{1}{\lim_{n\to\infty} a_{n-1}} &\implies r = 1 + \frac{1}{r} \\ &\implies 0 = r^2 - r - 1 \\ &\implies r = \frac{1+\sqrt{5}}{2} \text{ or } r = \frac{1-\sqrt{5}}{2}. \end{aligned}$$

It is clear that the negative root can't be the solution because all a_n are positive. Therefore,

$$\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}.$$

□

Solution 14: Let $a_0 = 0, a_1 = -1$, and for $n \geq 2,\ a_n - 7a_{n-1} + 12a_{n-2} = 0$.

Step 1: The associated characteristic equation is $r^2 - 7r + 12 = 0$ and its two roots are $r = 3$ and $r = 4$.

Step 2: We note that (i) the nonhomogeneous part is 0 and (ii) all characteristic roots are distinct. Thus, the general solution to the given recurrence equation is

$$a_n = A3^n + B4^n.$$

Step 3: We use the initial conditions to solve the following equations for the unknown constants A and B.

$$\left.\begin{array}{ll} n=0: & 0 = A + B, \\ n=1: & -1 = 3A + 4B. \end{array}\right\} \implies A = 1,\ B = -1.$$

Therefore,

$$a_n = 3^n - 4^n, n \geq 0.$$

□

Solution 15: Let $a_1 = 1, a_2 = 1$, and for $n \geq 3, a_n + 2a_{n-1} - 15a_{n-2} = 0$.

Step 1: The associated characteristic equation is $r^2 + 2r - 15 = 0$ and its two roots are $r = -5$ and $r = 3$.

Step 2: We note that (i) the nonhomogeneous part is 0 and (ii) all characteristic roots are distinct. Thus, the general solution is

$$a_n = A(-5)^n + B3^n.$$

Step 3: We use the initial conditions to solve the following equations for the unknown constants A and B. [Note: n starts from 1.]

$$\left.\begin{array}{ll} n=1: & 1 = -5A + 3B \\ n=2: & 1 = 25A + 9B \end{array}\right\} \implies A = -\frac{1}{20},\ B = \frac{1}{4}.$$

Therefore,

$$a_n = -\frac{1}{20} \times (-5)^n + \frac{1}{4} \times 3^n = \frac{1}{4}(-5)^{n-1} + \frac{1}{4} \times 3^n,\ n \geq 1.$$

□

Solution 16: Let $a_0 = 2, a_1 = 0$, and for $n \geq 2,\ -2a_n + 18a_{n-2} = 0$.

Step 1: The associated characteristic equation is $-2r^2 + 18 = 0$ and its two roots are $r = 3$ and $r = -3$.

Step 2: We note that (i) the nonhomogeneous part is 0 and (ii) all characteristic roots are distinct. Thus, the general solution is

$$a_n = A(-3)^n + B3^n.$$

Step 3: We use the initial conditions to solve the following equations for the unknown constants A and B.

$$\left.\begin{array}{ll} n=0: & 2 = \quad A + B \\ n=1: & 0 = -3A + 3B \end{array}\right\} \Longrightarrow A = 1,\ B = 1.$$

Therefore, $a_n = (-3)^n + 3^n,\ n \geq 0.$

□

Solution 17: Let $a_1 = 2, a_2 = 6$, and for $n \geq 3,\ \ a_n - 4a_{n-1} + 4a_{n-2} = 0.$

Step 1: The associated characteristic equation is $r^2 - 4r + 4 = 0$ and its two roots are $r = 2$ and $r = 2$.

Step 2: We have note that (i) the nonhomogeneous part is 0 and (ii) the two characteristic roots are the same. Thus, the general solution is

$$a_n = (A + Bn)2^n.$$

Step 3: We use the initial conditions to solve the following equations for the unknown constants A and B. [Note: n starts from 1.]

$$\begin{array}{ll} n=1: & 2 = \ (A+B)\times 2, \\ n=2: & 6 = (A+2B)\times 4. \end{array} \Longrightarrow A = \frac{1}{2},\ B = \frac{1}{2}.$$

Therefore,

$$a_n = (\frac{1}{2} + \frac{n}{2})2^n = (1+n)2^{n-1},\ n \geq 1.$$

□

Solution 18: Let $a_1 = 5, a_2 = -5$, and for $n \geq 3,\ a_n + 6a_{n-1} + 9a_{n-2} = 0.$

Step 1: The associated characteristic equation is $r^2 + 6r + 9 = 0$ and its two roots are $r = -3$ and $r = -3$.

Step 2: We have note that (i) the nonhomogeneous part is 0 and (ii) the two characteristic roots are the same. Thus, the general solution is

$$a_n = (A + Bn)(-3)^n.$$

Step 3: We use the initial conditions to solve the following equations for the unknown constants A and B. [Note: n starts from 1.]

$$\left.\begin{array}{rl} n=1: & 5 = (A+B)(-3) \\ n=2: & -5 = (A+2B)(-3)^2 \end{array}\right\} \Longrightarrow A = \frac{-25}{9},\ B = \frac{10}{9}.$$

Therefore,

$$a_n = (\frac{-25}{9} + \frac{10}{9}n)(-3)^n = 5(2n-5)(-3)^{n-2},\ n \geq 1.$$

□

Solution 19: Let $a_0 = 1, a_1 = 2$, and for $n \geq 2$, $a_n - 5a_{n-1} + 6a_{n-2} = 2n+1$.

This is a nonhomogeneous recurrence equation, hence its solution is obtained from two parts as shown in step 2 in the following.

Step 1: The associated characteristic equation is $r^2 - 5r + 6 = 0$ and its two roots are $r = 2$ and $r = 3$.

Step 2: (i) We note that the nonhomogeneous part is nonzero, which is

$$(2n+1) \cdot 1^n.$$

Since 1 is not one of the characteristic roots of the homogeneous part, therefore the solution to the nonhomogeneous part is a polynomial of degree of 1 (the degree of the nonhomogeneous part, $2n+1$). Let it be

$$Cn + D.$$

The solution to the nonhomogeneous part must also satisfy the difference equation. Thus,

$$(Cn + D) - 5(C(n-1) + D) + 6(C(n-2) + D) = 2n + 1.$$

After Simplifying, we obtain

$$2Cn + (2D - 7C) = 2n + 1.$$

By comparing the coefficients of n and the constant in the both sides of the above equality, we get

$$2C = 2,$$
$$2D - 7C = 1.$$

Therefore, $C = 1, D = 4$, and the solution to the nonhomogeneous part is

$$n + 4.$$

(ii) Since all characteristic roots are distinct, the general solution to the homogeneous part of the recurrence equation is $a_n = A2^n + B3^n$. By combining this solution with the solution to the nonhomogeneous part, $n + 4$, we get

$$a_n = A2^n + B3^n + n + 4.$$

Step 3: We use the initial conditions to solve the following equations for the unknown constants A and B.

$$\left.\begin{array}{l} n = 0: \ 1 = \quad A + B + 0 + 4, \\ n = 1: \ 2 = 2A + 3B + 1 + 4. \end{array}\right\} \Longrightarrow A = -6, \ B = 3.$$

Therefore,

$$a_n = -6 \cdot 2^n + 3 \cdot 3^n + n + 4 = -3 \cdot 2^{n+1} + 3^{n+1} + n + 4, \ n \geq 0.$$

□

Solution 20: Let $a_0 = 1, a_1 = -1$, and for $n \geq 2, \ a_n - 3a_{n-1} + 2a_{n-2} = n$.

As the previous problem, this is a nonhomogeneous recurrence equation. The solution is obtained from two parts.

Step 1: The associated characteristic equation is $r^2 - 3r + 2 = 0$ and its two roots are $r = 1$ and $r = 2$.

Step 2: (i) We note that the nonhomogeneous part is nonzero, which is $n \cdot 1^n$, and 1 is one of the characteristic roots. Thus, the solution to the nonhomogeneous part is a polynomial of degree 2. Let it be

$$Bn^2 + Cn + D.$$

Instead of finding the values of B, C, D at this point, we leave them unsolved until the final step.

(ii) By combining the result obtained in (i) with the contribution of the other characteristic root, $r = 2$, we get the general solution

$$a_n = A2^n + (Bn^2 + Cn + D)1^n.$$

Step 3: Since we have 4 unknown constants, A, B, C, and D, we need at least 4 initial values to solve them. We already have $a_0 = 1$ and $a_1 = -1$; two more are needed.

$$\begin{aligned} a_2 &= 3a_1 - 2a_0 + 2 &=& -3 - 2 + 2 = -3, \\ a_3 &= 3a_2 - 2a_1 + 3 &=& -9 + 2 + 3 = -4. \end{aligned}$$

These four initial conditions give the following linear equations:

$$\begin{aligned} n = 0: \quad 1 &= A + 0 + 0 + D, \\ n = 1: \quad -1 &= 2A + B + C + D, \\ n = 2: \quad -3 &= 4A + 4B + 2C + D, \\ n = 3: \quad -4 &= 8A + 9B + 3C + D. \end{aligned}$$

A typical systematic method to solve the above system is by reducing the matrix of coefficients to an upper triangular matrix via row and column operations as shown below.[2]

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & -1 \\ 4 & 4 & 2 & 1 & -3 \\ 8 & 9 & 3 & 1 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 4 & 2 & -3 & -7 \\ 0 & 9 & 3 & -7 & -12 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & -2 & 1 & 5 \\ 0 & 0 & -6 & 2 & 15 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & -2 & 1 & 5 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Using the last matrix and working backwards give the following four equations, which are very easy to solve.

$$\begin{aligned} -D = 0 &\Longrightarrow D = 0, \\ -2C + D = 5 &\Longrightarrow C = -\frac{5}{2}, \\ B + C - D = -3 &\Longrightarrow B = -\frac{1}{2}, \\ A + D = 1 &\Longrightarrow A = 1. \end{aligned}$$

Therefore,

$$a_n = 2^n - \frac{n^2 + 5n}{2}, \;\; n \geq 0.$$

□

[2] One can refer to any textbook in linear algebra for more details.

Solution 21: Given $a_0 = 0$, $a_1 = 1$, $a_2 = 4$, $a_3 = 13$, and

$$a_n + ba_{n-1} + ca_{n-2} = 0 \quad \text{for } n \geq 2,$$

We use the given initial values to solve b and c. We know that

$$\begin{aligned} a_3 + ba_2 + ca_1 &= 0, \\ a_2 + ba_1 + ca_0 &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} 13 + 4b + c &= 0, \\ 4 + b + 0 &= 0. \end{aligned}$$

By solving the above equations, we get $b = -4, c = 3$. Then, we have a standard difference equation and sufficient information to solve it. $a_0 = 0$, $a_1 = 1$, and

$$a_n - 4a_{n-1} + 3a_{n-2} = 0 \quad \text{for } n \geq 2,$$

Step 1: The associated characteristic equation is $r^2 - 4r + 3 = 0$ and its two roots are $r = 1$ and $r = 3$.

Step 2: We note that (i) the nonhomogeneous part is 0 and (ii) all characteristic roots are distinct. Thus, the general solution is

$$a_n = A3^n + B1^n.$$

Step 3: We use the initial conditions to solve the following equations for the unknown constants A and B.

$$\left.\begin{aligned} n = 0: \ 0 &= \ \ A + B \\ n = 1: \ 1 &= 3A + B \end{aligned}\right\} \Longrightarrow A = \frac{1}{2}, \ B = -\frac{1}{2}.$$

Therefore,

$$a_n = \frac{1}{2}3^n - \frac{1}{2}, \ n \geq 0.$$

□

Solution 22: Let $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$, $3a_n - 10a_{n-1} + 3a_{n-2} = 3^n$.

This is a nonhomogeneous recurrence equation. The solution is obtained from two parts.

Step 1: The associated characteristic equation is $3r^2 - 10r + 3 = 0$ and its two roots are $r = \frac{1}{3}$ and $r = 3$.

Step 2: (i) We note that the nonhomogeneous part is nonzero, which is 3^n, and 3 is one of the characteristic roots. Thus, the solution to the nonhomogeneous part is the product of 3^n and a polynomial of degree 1. Let it be

$$(Bn + C)3^n.$$

(ii) By combining the result obtained in (i) with the contribution of the other characteristic root, $r = \frac{1}{3}$, we get the general solution

$$a_n = A(\frac{1}{3})^n + (Bn + C)3^n.$$

Step 3: Since we have three unknown constants A, B and C, we need at least 3 initial values to solve them. We already have $a_0 = 1$ and $a_1 = 1$; a_2 can be obtained from the given difference equation.

$$3a_2 = 3^2 + 10a_1 - 3a_0 = 9 + 10 \implies a_2 = \frac{19}{3}.$$

Now A, B, and C can be determined from the following system:

$$\begin{aligned}
n = 0: \quad 0 &= A + 0 + C, \\
n = 1: \quad 1 &= \tfrac{1}{3}A + 3B + 3C, \\
n = 2: \quad \tfrac{19}{3} &= \tfrac{1}{9}A + 18B + 9C.
\end{aligned}$$

By solving the system, we get

$$A = \frac{3}{64},\ B = \frac{3}{8},\ C = -\frac{3}{64}.$$

Therefore,

$$a_n = \frac{3}{64} \cdot (\frac{1}{3})^n + (\frac{3}{8} \cdot n - \frac{3}{64})3^n,\ n \geq 0.$$

□

Solution 23: Let $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$, $5a_n - 6a_{n-1} + a_{n-2} = n^2(\frac{1}{5})^n$.

This is a nonhomogeneous recurrence equation. The solution is obtained from two parts.

Step 1: The associated characteristic equation is $5r^2 - 6r + 1 = 0$ and its two roots are $r = \frac{1}{5}$ and $r = 1$.

Step 2: (i) We note that the nonhomogeneous part is

$$n^2(\frac{1}{5})^n,$$

and $\frac{1}{5}$ is one of the characteristic roots, thus the solution to the nonhomogeneous part is the product of $(\frac{1}{5})^n$ and a polynomial of degree 3 (the degree of n^2 plus 1). Let it be

$$(An^3 + Bn^2 + Cn + D)(\frac{1}{5})^n.$$

(ii) The other characteristic root is 1. Combining with the result in (i), the general solution is

$$a_n = (An^3 + Bn^2 + Cn + D)(\frac{1}{5})^n + E \cdot 1^n.$$

Step 3: Since we have 5 unknown constants, we need at least 5 initial values to solve them. We already have $a_0 = 1$ and $a_1 = 1$; and we will generate three more by using the given difference equation and a_0 and a_1. We get

$$a_2 = \frac{154}{5^3},\ a_3 = \frac{808}{5^4},\ a_4 = \frac{4094}{5^5}.$$

We have the following system:

$$\begin{array}{ccccccccccc} 0 & = & 0 & + & 0 & + & 0 & + & D & + & E \\ 1 & = & A & + & B & + & C & + & D & + & E \\ \frac{154}{5^3} & = & \frac{8A}{5^2} & + & \frac{4B}{5^2} & + & \frac{2C}{5^2} & + & \frac{D}{5^2} & + & E \\ \frac{808}{5^4} & = & \frac{27A}{5^3} & + & \frac{9B}{5^3} & + & \frac{3C}{5^3} & + & \frac{D}{5^3} & + & E \\ \frac{4094}{5^5} & = & \frac{64A}{5^4} & + & \frac{16B}{5^4} & + & \frac{4C}{5^4} & + & \frac{D}{5^4} & + & E \end{array}$$

After solving the system, we get

$$A = -\frac{97}{10560},\ B = -\frac{387}{3520},\ C = -\frac{79}{528},\ D = -\frac{843}{640},\ E = \frac{843}{640}.$$

□