Problems on Discrete Mathematics[1]

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Chapter 2

Logic

It presupposes nothing but logic; that is,
logic is the only preceding theory.

– Alfred Tarski

2.1 Definitions

2.1.1 Propositional Logic

Definition 2.1: *Propositions* are mathematical statements such that their truth or falsity can be told without ambiguity. A proposition is also called a primitive statement.

We use letters $p, q, r, \ldots$ to denote propositions. Thus, the values of $p, q, r, \ldots$ are either $True$ or $False$.

Example 2.1

$$\begin{array}{ll} p: & 2+2=5, \\ q: & x^2-2x+1 \text{ has two identical roots}, \\ r: & \emptyset \subset \{1,2,\emptyset\}, \\ s: & \text{there exists a 100-digit prime number}, \end{array}$$

are propositions. The value of p is $False$, and the values of q and r are $True$. We will use T to denote $True$ and F to denote $False$ for the rest of this book.

Comment: The statement s in the above example is a proposition, even though we do not know for sure that there exists a 100-digit prime number, but we are sure that s is either T or F. However, not every mathematical statement has a truth value. The following two statements are not propositions.

$$\begin{array}{ll} u: & \text{This statement is false.} \\ v: & a \in S, \text{ where } S \text{ is a set defined as } S = \{x|x \notin S\}. \end{array}$$

Definition 2.2: Suppose $p, q, r, \ldots$ are variables with values either T or F. We call such variables *propositional variables*. We can assign T or F to any propositional variable as we wish. propositional variables are also known as *atoms*.

Definition 2.3: Let p, q be two propositions. We use the symbols:

$$\neg,\ \wedge,\ \vee,\ \rightarrow,\ \leftrightarrow$$

to construct new propositions such as

$$\neg p,\ p \wedge q,\ p \vee q,\ p \rightarrow q,\ p \leftrightarrow q..$$

These symbols are called *logical connectives*; $\neg$ is read as "not" or "negation", $\wedge$ as "and" or "conjunction", $\vee$ as "or" or "disjunction", $p \rightarrow q$ is read as "if p then q" or "p implies q", and $p \leftrightarrow q$ is read as "p if and only if q".

Recall that all propositions have truth values either T or F. We decide the values of the new propositions constructed above based on the values of p and q and the rules of logical connectives defined below.

Definition 2.4: 1. The negation of p: $\neg p$.

p	$\neg p$
T	F
F	T

2. The conjunction of p and q: $p \wedge q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. The disjunction of p and q: $p \vee q$.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4. Implication, if p then q: $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

5. Equivalence, p if and only if q: $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 2.5: Given $p \rightarrow q$, we call $\neg q \rightarrow \neg p$ the *contrapositive*, call $q \rightarrow p$ the *converse*, and call $\neg p \rightarrow \neg q$ the *inverse* of $p \rightarrow q$.

Definition 2.6: *Propositional formulas* are recursively defined as follows:

1. T and F are propositional formulas.
2. All atoms are propositional formulas.
3. If f_1 and f_2 are propositional formulas, so are

$$\neg f_1,\ f_1 \wedge f_2,\ f_1 \vee f_2,\ f_1 \rightarrow f_2,\ f_1 \leftrightarrow f_2.$$

Only the formulas generated by the rules above are propositional formulas. Propositional formulas are also known as *well-formed-formulas*, or wff's in short.

Example 2.2

$$\begin{array}{llll} pq\wedge, & \neg\vee pq, & \rightarrow(p\wedge q)\vee r & \text{are not wff's.} \\ p\wedge q, & \neg p\vee q, & (p\wedge q)\rightarrow r & \text{are wff's.} \end{array}$$

We are allowed to use parentheses with their natural purposes.

Definition 2.7: The operational priorities of logical connectives are defined below:

Precedence : $\{\neg\} > \{\wedge, \vee\} > \{\rightarrow, \leftrightarrow\}$.
In other words, "$\neg$" is evaluated before "$\wedge$" and "$\vee$", and "$\wedge$" and "$\vee$" are evaluated before "$\rightarrow$" and "$\leftrightarrow$". For example,

$$a \rightarrow b \wedge \neg c \;\equiv\; a \rightarrow (b \wedge (\neg c)).$$

The equivalence symbol $\equiv$ above means: $a \rightarrow b\wedge\neg c$ is to be interpreted as $a \rightarrow (b\wedge(\neg c))$, and $a \rightarrow (b\wedge(\neg c))$ can be abbreviated as $a \rightarrow b\wedge\neg c$. We can alternatively use one of them without introducing ambiguity.

Associativity : $\wedge$ and $\vee$ are *left associative*; $\rightarrow$ and $\leftrightarrow$ are *right associative*. For example,

$$a\wedge b\vee c \;\equiv (a\wedge b)\vee c,$$
$$a\rightarrow b\leftrightarrow c\rightarrow d \;\equiv a\rightarrow(b\leftrightarrow(c\rightarrow d)).$$

Definition 2.8: Let f be a wff in n variables. The *truth table* of f is a table that contains all possible values of the variables in the rows (each row represents one possibility), and the corresponding values of f are in the last column. The tables in definition 2.4 are the truth tables of $\neg p$, $p\wedge q$, $p\vee q$, $p\rightarrow q$, and $p\rightarrow q$, respectively. For complicated formulas, we can add some intermediate columns to help us find the truth values of f.

Example 2.3 Let $f = (p\wedge q)\rightarrow r$. The truth table of f is

p	q	r	$p\wedge q$	$(p\wedge q)\rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

We have added a column to present the truth values of $p\wedge q$. The 3rd row, for example, indicates that if $p = T, q = F$, and $r = T$, then f is T.

Definition 2.9: Let f be a wff. f is called a *tautology* if and only if f is T everywhere in the last column of its truth table.

Definition 2.10: Let f be a wff. f is called a *contradiction* if and only if f is F everywhere in the last column of its truth table.

Definition 2.11: Let f_1 and f_2 be two wff's. Define

$$f_1 \Rightarrow f_2 \text{ if and only if } f_1 \rightarrow f_2 \text{ is a tautology.}$$

We say f_1 *logically implies* f_2.

Definition 2.12: Let f_1 and f_2 be two wff's. Define

$$f_1 \iff f_2 \text{ if and only if } f_1 \leftrightarrow f_2 \text{ is a tautology.}$$

We say f_1 and f_2 are *logically equivalent*.

Definition 2.13: Let f be a wff. If the logical connectives that f contains are $\wedge$ and $\vee$ only, then the dual of f, denoted as f^d, is a wff obtained from f by the following rules: In f,

1. replace T by F, and F by T,
2. replace $\wedge$ by $\vee$,
3. replace $\vee$ by $\wedge$.

Definition 2.14: Let f be a wff in n variables. The Disjunctive Normal Form, DNF in short, of f is a logical equivalence of f, which is a disjunction of one or more distinct $(x_1 \wedge x_2 \wedge \cdots \wedge x_n)$, where $x_i, 1 \leq i \leq n$, is a propositional variable of f or its negation. Each x_i is known as a literal.

Example 2.4 Let $f = p \rightarrow q$. The DNF of f is

$$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q). \tag{2.1}$$

Because, (a) both f and the proposition (1) have the same truth values for all possible values of p and q, and (b) the proposition (1) is a disjunction of distinct formulas using the connective "$\wedge$" between literals.

Definition 2.15: Let f be a wff in n variables. The Conjunctive Normal Form, CNF in short, of f is a logical equivalence of f, which is a conjunction of one or more distinct $(x_1 \vee x_2 \vee \cdots \vee x_n)$, where $x_i, 1 \leq i \leq n$, is a literal.

Example 2.5 Let $f := p \rightarrow q$. The CNF of f is

$$(\neg p \vee q).$$

2.1.2 Predicate Logic

Definition 2.16: A *predicate* $P(x_1, x_2, \cdots, x_n), n \geq 0$ is a mapping from the concerned domains to $\{T, F\}$. In other words, if we replace $x_1, \cdots, x_n$ by instances in their corresponding domains, we will get a proposition.

Example 2.6 The following are all predicates.

$$\begin{array}{lll} P(x,y): & x \geq y^2, & D_x = D_y = \mathbf{N}; \\ Q(x,y): & x \text{ is a } y, & D_x \text{ is the set of names,} \\ & & D_y = \{\text{father, son, cat, dog, table,} \ldots\} \end{array}$$

We use D_x, D_y to denote the domains of x and y, respectively. Note that D_x and D_y do not have to be the same. In the above example, $P(3, 2)$ is the proposition $3 \geq 2^2$ with truth value F. Similarly, $Q(\text{Boo}, \text{dog})$ is a proposition with truth value T if there is a dog named Boo.

Note: Any proposition is a predicate in zero variables.

Definition 2.17: Let $P(x_1, x_2, \cdots, x_n), n \geq 0$ be a predicate. The set $D_{x_1} \times D_{x_2} \times \cdots \times D_{x_n}$ is called the *domain of predicate* P.

Definition 2.18: Let $P(x_1, x_2, \cdots, x_n), n \geq 0$ be a predicate. The *truth set* of P, denoted as T_P, is a subset of $D_{x_1} \times D_{x_2} \times \cdots \times D_{x_n}$ such that for all $(a_1, a_2, \ldots, a_n) \in T_P$, $P(a_1, a_2, \ldots, a_n) = T$.

Definition 2.19: Let $P(x_1, x_2, \cdots, x_n), n \geq 0$ be a predicate. The *falsity set* of P, denoted as F_P, is a subset of $D_{x_1} \times D_{x_2} \times \cdots \times D_{x_n}$ such that for all $(a_1, a_2, \ldots, a_n) \in F_P$, $P(a_1, a_2, \ldots, a_n) = F$.

Definition 2.20: Let P, Q be two predicates. We say that P and Q are logically equivalent if and only if $T_p = T_Q$.

Definition 2.21: Let P, Q be two predicates. We say that P logically implies Q if and only if $T_p \subseteq T_Q$.

In addition to the connectives discussed in the previous section, we have two special symbols called quantifiers in predicate logic. The two special symbols are $\forall$ and $\exists$. The quantifiers $\forall$ and $\exists$ are called the *universal quantifier* and the *existential quantifier* and pronounced as "for all" and "there exists" respectively. The meaning of $\forall$ and $\exists$ are defined below.

Definition 2.22: Let $P(x)$ be a predicate in one variable.

1. Universal quantifier: $\forall x \in D_x P(x)$ is a proposition, and its value is T if every instance a in D_x makes $P(a)$ true.
2. Existential quantifier: $\exists x \in D_x P(x)$ is a proposition, and its value is T if there exists some instance a in D_x that makes $P(a)$ true.

If the domain D_x is clear from the context, we usually drop the domain set and rewrite $\forall x \in D_x P(x)$ as $\forall x P(x)$ and $\exists x \in D_x P(x)$ as $\exists x P(x)$. A predicate preceded with one or more quantifiers is called a *quantified predicate*.

Definition 2.23: Let $P(x_1, \ldots, x_i, \ldots, x_n)$ be a predicate in n variables. Then, $\forall x_i P(x_1, \ldots, x_i, \ldots, x_n)$ and $\exists x_i P(x_1, \ldots, x_i, \ldots, x_n)$ are predicates in $n-1$ variables, where x_i is called a *bounded variable* and the rest of the variables are called *unbounded variables* or *free variables*.

Example 2.7 Given the predicate $P(x, y)$ in two variables x and y, $\forall x P(x, y)$ and $\exists x P(x, y)$ are predicates in one variable y. The variable y is a free variable, and x is a bounded variable in $\forall x P(x, y)$ and $\exists x P(x, y)$.

2.1.3 Predicates and Sets

Let $P(x)$ and $Q(x)$ be two predicates in one variable, and let D_x be the universe of x. Then the associated truth and falsity sets satisfy the following properties.

1. $T_P \cup F_P = D_x$.
2. $T_P \cap F_P = \emptyset$.
3. $T_{P \wedge Q} = T_P \cap T_Q$.
4. $T_{P \vee Q} = T_P \cup T_Q$.
5. $\forall x[P(x) \rightarrow Q(x)] \iff T_P \subseteq T_Q$.
6. $\exists x[P(x) \wedge Q(x)] \iff (T_P \cap T_Q) \neq \emptyset$.

2.2 Logical Proof

Logical proof is a formal way of convincing that some statements are correct based on given facts.

Let P and Q be two wff's, predicates, or quantified predicates. Starting from P, if we can find a sequence of applications of laws of logic, rules of inference, or tautologies to arrive at Q, step by step, we say there is a proof for the *theorem*

$$P \Longrightarrow Q.$$

In theorem $P \Rightarrow Q$, P is called the *premise*, and Q is called a *logical conclusion* of P. The sequence of these steps is called a *logical proof* of $P \Rightarrow Q$. A logical proof is usually represented in a table like the following one.

Premises: P
Conclusion: Q

	steps	*reasons*
1.	P	Premises
2.	q_2	$--$
3.	q_3	$--$
$\vdots$	$\vdots$	$\vdots$
$i.$	q_i	$--$
$\vdots$	$\vdots$	$\vdots$
$n.$	Q	$--$

where $--$ are the names of the logical rules.

In short, a logical proof can be written as

$$P \Rightarrow q_2 \Rightarrow q_3 \Rightarrow \cdots \Rightarrow q_i \Rightarrow \ldots \Rightarrow Q.$$

P and Q are not necessarily two single statements; they can be a set of wff's, predicates, and quantified predicates. For example, a theorem may look like

$$\{P_1, P_2, \ldots, P_n\} \Longrightarrow \{Q_1, Q_2, \ldots, Q_m\}. \tag{2.2}$$

On the other hand, a punctuation symbol "," is considered as a "$\wedge$". Therefore, the theorem in (2) can be expressed as

$$(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \Longrightarrow (Q_1 \wedge Q_2 \wedge \ldots \wedge Q_m).$$

2.2.1 Laws of Logic

Let $f_1 \leftrightarrow f_2$ be a tautology, i.e., $f_1 \Leftrightarrow f_2$. During the course of deriving the sequence of a logical proof, if one of f_1 and f_2 appears, then we can attach the other one to the sequence of the logical proof.

For example, if we know that $f_1 \Leftrightarrow f_2$ and already have

$$P \Rightarrow q_2 \Rightarrow q_3 \Rightarrow \cdots \Rightarrow f_1,$$

then the proof above can be extended to

$$P \Rightarrow q_2 \Rightarrow q_3 \Rightarrow \cdots \Rightarrow f_1 \Rightarrow f_2.$$

The following is a list of tautologies, where p, q, and r are wff's. One can use the definitions of connectives to prove that they are tautologies. We call them *Laws of Logic*. In a logical proof we are allowed to use the laws of logic directly without further proof.

1. Law of double negation:
$$\neg\neg p \iff p.$$

2. Absorption Laws:
$$p \wedge (p \vee q) \iff p,$$
$$p \vee (p \wedge q) \iff p.$$

3. Idempotent Laws:
$$p \wedge p \iff p,$$
$$p \vee p \iff p.$$

4. Inverse Laws:
$$p \wedge \neg p \iff F,$$
$$p \vee \neg p \iff T.$$

5. Identity Laws:
$$p \wedge T \iff p,$$
$$p \vee F \iff p.$$

6. Domination Laws:
$$p \wedge F \iff F,$$
$$p \vee T \iff T.$$

7. Commutative Laws:
$$(p \wedge q) \iff (q \wedge p),$$
$$(p \vee q) \iff (q \vee p).$$

8. Associative Laws:
$$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r),$$
$$(p \vee (q \vee r)) \iff ((p \vee q) \wedge r).$$

9. Distributive Laws:
$$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r)),$$
$$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r)).$$

10. Contrapositive Law:
$$(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$$

11. De Morgan's Laws:
$$\neg(p \vee q) \iff (\neg p \wedge \neg q),$$
$$\neg(p \wedge q) \iff (\neg p \vee \neg q).$$

12. No specific name is given, but this law is one of the most frequently used laws in logical proof.
$$(p \rightarrow q) \iff (\neg p \vee q).$$

2.2.2 Rules of Inference

Let $f_1 \to f_2$ be a tautology, i.e., $f_1 \Rightarrow f_2$. During the course of deriving the sequence of a logical proof, if f_1 appears, then we can attach f_2 to the sequence of the logical proof.

For example, if we know that $f_1 \Rightarrow f_2$, and already have

$$P \Rightarrow q_2 \Rightarrow q_3 \Rightarrow \cdots \Rightarrow f_1,$$

then the proof above can be extended to

$$P \Rightarrow q_2 \Rightarrow q_3 \Rightarrow \cdots \Rightarrow f_1 \Rightarrow f_2.$$

Let p, q, r, and s be wff's. The following tautologies are called *Rules of Inference*.

1. Modus Ponens:
$$(p \wedge (p \to q)) \Longrightarrow q.$$

2. Modus Tollens:
$$(\neg q \wedge (p \to q)) \Longrightarrow \neg p.$$

3. Law of the Syllogism:
$$((p \to q) \wedge (q \to r)) \Longrightarrow (p \to r).$$

4. Rule of Disjunctive Syllogism:
$$((p \vee q) \wedge \neg q) \Longrightarrow p.$$

5. Rule of Contradiction:
$$(\neg p \to F) \Longrightarrow p.$$

6. Rule of Conjunctive Simplification:
$$(p \wedge q) \Longrightarrow p.$$

7. Rule of Disjunctive Amplification:
$$p \Longrightarrow (p \vee q).$$

8. Proof by cases:
$$((p \to r) \wedge (q \to r)) \Rightarrow ((p \vee q) \to r)$$

2.2.3 Inference Rules for Quantified Predicates

Let $P(x)$ and $Q(x, y)$ be two predicates in one and two variables respectively. Following are some basic inference rules for quantified predicates.

1. Negation:
$$\neg\forall x P(x) \Longleftrightarrow \exists x \neg P(x),$$
$$\neg\exists x P(x) \Longleftrightarrow \forall x \neg p(x).$$

2. α-Conversion (changing the name of the bounded variable):
$$\forall x P(x) \Longleftrightarrow \forall y P(y),$$
$$\exists x P(x) \Longleftrightarrow \exists y P(y).$$

3. Reordering quantifiers of the same kind:
$$\forall x \forall y Q(x, y) \Longleftrightarrow \forall y \forall x Q(x, y),$$
$$\exists x \exists y Q(x, y) \Longleftrightarrow \exists y \exists x Q(x, y).$$

4. Universal Specification:
$$\forall x P(x) \Longrightarrow P(a), \text{ any } a \in D_x.$$

5. Existential Specification:
$$\exists x P(x) \Longrightarrow P(a), \text{ some } a \in D_x.$$

6. Universal Generalization:
$$(\text{any } a \in D_x, P(a) = T) \Longrightarrow \forall x P(x).$$

7. Existential Generalization:
$$(\text{some } a \in D_x, P(a) = T) \Longrightarrow \exists x P(x).$$

Comment: We do not have a standard notation to distinguish a between "any a" in 6 and "some a" in 7. One should find a way to make them clear in the proof without confusion. See problem 50 of this chapter.

2.3 DNF and CNF

In this section we introduce a systematic way to find the DNF and CNF of any given wff. The DNF and CNF are perhaps not mathematically dignified for presenting any wff, but they are essentially the underlying presentation inside modern digital computers. The DNF and CNF provide a convenient apparatus in applications of Artificial Intelligence, Logical Programming, and many other areas of research. Therefore, it behooves us to pay more attention to the DNF and CNF.

We will pedagogically discuss wff's in three variables. One can easily extend the method to a more general case.

Definition 2.24: Let f be a wff in variables x_1, x_2 and x_3, and let $i \in \{1, 2, 3\}$.

1. Each term x_i or its complement $\neg x_i$ is called a *literal.*
2. A term of the form $y_1 \wedge y_2 \wedge y_3$, where $y_i = x_i$ or $y_i = \neg x_i$ is called a *fundamental conjunction.*
3. A term of the form $y_1 \vee y_2 \vee y_3$, where $y_i = x_i$ or $y_i = \neg x_i$ is called a *fundamental disjunction.*
4. A representation of f in a disjunction of fundamental conjunctions is called a *disjunctive normal form* (DNF) or *sum-of-product* from.
5. A representation of f in a conjunction of fundamental disjunctions is called a *conjunctive normal form* (CNF) or *product-of-sum* from.

2.3.1 The DNF of a given wff

$$\text{DNF:} \quad D_1 \vee D_2 \vee \cdots \vee D_n. \tag{2.3}$$

Let f be a wff in three variables a, b, and c. There are eight possible fundamental conjunctions for a D in (3). These fundamental conjunctions are also known as the building blocks for f in DNF. Let d_i be the i^{th} building block defined in the follows.

$$\left.\begin{array}{ll} d_1: & a \wedge b \wedge c \\ d_2: & a \wedge b \wedge \overline{c} \\ d_3: & a \wedge \overline{b} \wedge c \\ d_4: & a \wedge \overline{b} \wedge \overline{c} \\ d_5: & \overline{a} \wedge b \wedge c \\ d_6: & \overline{a} \wedge b \wedge \overline{c} \\ d_7: & \overline{a} \wedge \overline{b} \wedge c \\ d_8: & \overline{a} \wedge \overline{b} \wedge \overline{c} \end{array}\right\}\text{Building Blocks for DNF in three variables.}$$

We use $\overline{x}$ to denote $\neg x$ due to space consideration.

Step 1: Evaluate the truth value of each building block. It is convenient to use a truth table as shown below, where the truth value associated with each building block is evaluated and followed by the truth value of f.

a	b	c	$a \wedge b \wedge c$	$a \wedge b \wedge \bar{c}$	$a \wedge \bar{b} \wedge c$	$\cdots$	$\cdots$	$f(a,b,c)$	
T	T	T	T ✓	F	F	$\cdots$	$\cdots$	T	✓
T	T	F	F	T	F	$\cdots$	$\cdots$	F	
T	F	T	F	F	T ✓	$\cdots$	$\cdots$	T	✓
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\ddots$	$\vdots$	

Step 2: We check the last column and mark the rows in which f's value is T. For each marked row, we find and mark the building block with value T in the row. Note that for each row, there is exactly one building block with value T, and for each column of a building block, there is exactly one row with value T.

Step 3: Finally, the DNF is the disjunction of the building blocks marked in step 2. In the above partial example, a part of the DNF is

$$f(a,b,c) = (a \wedge b \wedge c) \vee (a \wedge \bar{b} \wedge c) \vee \cdots$$

□

Example 2.8 Find the DNF of $(a \longrightarrow b) \vee c$.

Let $f(a,b,c) = (a \longrightarrow b) \vee c$, and $d_1, d_2, \ldots, d_8$ be the building blocks we just defined. Follow the rules we just described step by step, we have the following table.

a	b	c	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	f	
T	T	T	T ✓	F	F	F	F	F	F	F	T	✓
T	T	F	F	T ✓	F	F	F	F	F	F	T	✓
T	F	T	F	F	T ✓	F	F	F	F	F	T	✓
T	F	F	F	F	F	T	F	F	F	F	F	
F	T	T	F	F	F	F	T ✓	F	F	F	T	✓
F	T	F	F	F	F	F	F	T ✓	F	F	T	✓
F	F	T	F	F	F	F	F	F	T ✓	F	T	✓
F	F	F	F	F	F	F	F	F	F	T ✓	T	✓

According to the rule in step 3, the DNF of f is

$$d_1 \vee d_2 \vee d_3 \vee d_5 \vee d_6 \vee d_7 \vee d_8.$$

I.e.,

$$\begin{aligned}(a \rightarrow b) \vee c \quad = \quad & (a \wedge b \wedge c) \vee (a \wedge b \wedge \bar{c}) \vee (a \wedge \bar{b} \wedge c) \vee (\bar{a} \wedge b \wedge c) \vee \\ & (\bar{a} \wedge b \wedge \bar{c}) \vee (\bar{a} \wedge \bar{b} \wedge c) \vee (\bar{a} \wedge \bar{b} \wedge \bar{c}).\end{aligned}$$

□

2.3.2 The CNF of a given wff

$$\text{CNF:} \quad C_1 \wedge C_2 \wedge \cdots \wedge C_n.$$

Let f be a wff in 3 variables a, b, and c. There are 8 building blocks for f in CNF. Let c_i be the i^{th} building block defined in the follows.

$$\left.\begin{array}{rl} c_1: & a \vee b \vee c \\ c_2: & a \vee b \vee \overline{c} \\ c_3: & a \vee \overline{b} \vee c \\ c_4: & a \vee \overline{b} \vee \overline{c} \\ c_5: & \overline{a} \vee b \vee c \\ c_6: & \overline{a} \vee b \vee \overline{c} \\ c_7: & \overline{a} \vee \overline{b} \vee c \\ c_8: & \overline{a} \vee \overline{b} \vee \overline{c} \end{array}\right\} \text{Building Blocks for CNF in 3 variables.}$$

Step 1: We construct the following truth table. As we did for finding DNF, each building block must occupy one column, and the last column contains the truth values of f.

a	b	c	$a \vee b \vee c$	$a \vee b \vee \overline{c}$	$\cdots\cdots$	$\overline{a} \vee \overline{b} \vee c$	$\cdots$	$f(a,b,c)$	
T	T	T	T	T	$\cdots\cdots$	T	$\cdots$	T	
T	T	F	T	T	$\cdots\cdots$	F ✓	$\cdots$	F	✓
T	F	T	T	T	$\cdots\cdots$	T	$\cdots$	T	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	
F	F	F	F ✓	T	$\cdots\cdots$	T	$\cdots$	F	✓

Step 2: Check the last column and mark the rows in which f's value is F. For each marked row, we find and mark the building block with value F in the row. Note that for each row, there is exactly one building block with value F, and there is exactly one row with value F in each building block's column.

Step 3: Finally, the CNF is the conjunction of the building blocks marked in step 2. In the above partial example, a part of the CNF is

$$f(a,b,c) = (a \vee b \vee c) \ \wedge \ (\overline{a} \vee \overline{b} \vee c) \ \wedge \ \cdots$$

□

Example 2.9 Find the CNF of $(a \longrightarrow b) \vee c$.

Let, $f(a, b, c) = (a \longrightarrow b) \vee c$, and $c_1, c_2, \ldots, c_8$ be the building blocks we just defined.

a	b	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	f	
T	T	T	T	T	T	T	T	T	T	F	T	
T	T	F	T	T	T	T	T	T	F	T	T	
T	F	T	T	T	T	T	T	F	T	T	T	
T	F	F	T	T	T	T	F √	T	T	T	F	√
F	T	T	T	T	T	F	T	T	T	T	T	
F	T	F	T	T	F	T	T	T	T	T	T	
F	F	T	T	F	T	T	T	T	T	T	T	
F	F	F	F	T	T	T	T	T	T	T	T	

Therefore,

$$(a \longrightarrow b) \vee c = (\overline{a} \vee b \vee c).$$

□

Example 2.10 Find the DNF and CNF of $a \wedge (b \longleftrightarrow c)$.

Let $f = a \wedge (b \longleftrightarrow c)$, and $d_1, d_2, \ldots, d_8$ be the building blocks for DNF and $c_1, c_2, \ldots, c_8$ the building blocks for CNF as defined.

For DNF:

a	b	c	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	f	
T	T	T	T √	F	F	F	F	F	F	F	T	√
T	T	F	F	T	F	F	F	F	F	F	F	
T	F	T	F	F	T	F	F	F	F	F	F	
T	F	F	F	F	F	T √	F	F	F	F	T	√
F	T	T	F	F	F	F	T	F	F	F	F	
F	T	F	F	F	F	F	F	T	F	F	F	
F	F	T	F	F	F	F	F	F	T	F	F	
F	F	F	F	F	F	F	F	F	F	T	F	

Therefore, the DNF of f is $d_1 \wedge d_4$, i.e.,

$$(a \wedge b \wedge c) \vee (a \wedge \overline{b} \wedge \overline{c}).$$

For CNF:

a	b	c	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	f	
T	T	T	T	T	T	T	T	T	T	F	T	
T	T	F	T	T	T	T	T	T	F √	T	F	√
T	F	T	T	T	T	T	T	F √	T	T	F	√
T	F	F	T	T	T	T	F	T	T	T	T	
F	T	T	T	T	T	F √	T	T	T	T	F	√
F	T	F	T	T	F √	T	T	T	T	T	F	√
F	F	T	T	F √	T	T	T	T	T	T	F	√
F	F	F	F √	T	T	T	T	T	T	T	F	√

Therefore, the CNF of f is $c_1 \wedge c_2 \wedge c_3 \wedge c_4 \wedge c_6 \wedge c_7$, i.e.,

$$(a \vee b \vee c) \wedge (a \vee b \vee \bar{c}) \wedge (a \vee \bar{b} \vee c) \wedge (a \vee \bar{b} \vee \bar{c}) \wedge (\bar{a} \vee b \vee \bar{c}) \wedge (\bar{a} \vee \bar{b} \vee c).$$

□

2.3.3 A shortcut to find the DNF and CNF

Given any wff f, we can find the DNF and CNF directly from its truth set and falsity set. Let us assume that f has three variables a, b, and c, and let the format of the truth set T_f and the falsity set F_f be $D_a \times D_b \times D_c$.

Step 1: Find out the truth set T_f and the falsity set F_f of f.

Step 2: For DNF, we will use the truth set T_f. The building blocks that should appear in the DNF of f are those having truth value T if we apply some elements in T_f to them. In other words, for each $(t_a, t_b, t_c) \in T_f$, we choose $(x_a \wedge x_b \wedge x_c)$ according to the following rules.

1. $x_a = a$ if $t_a = T$.
2. $x_a = \neg a$ if $t_a = F$.
3. x_b and x_c are decided by the same principle above.

For CNF, we will use the falsity set F_f. The building blocks that should appear in the CNF of f are those having truth value F if we apply some elements in F_f to them. In other words, for each $(t_a, t_b, t_c) \in F_f$, we choose $(x_a \wedge x_b \wedge x_c)$ according to the following rules.

1. $x_a = \neg a$ if $t_a = T$.
2. $x_a = a$ if $t_a = F$.
3. x_b and x_c are decided by the same principle above.

Basically, the shortcut method and the truth table addressed with building blocks are essentially the same. Let's consider the following example.

Example 2.11 Find the DNF and CNF of $(a \leftrightarrow b) \leftrightarrow c$ from its truth set and falsity set directly.

Let $f = (a \leftrightarrow b) \leftrightarrow c$. We first find T_f and F_f by using the following truth table.

a	b	c	$a \leftrightarrow b$	$(a \leftrightarrow b) \leftrightarrow c$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	T
F	T	T	F	F
F	T	F	F	T
F	F	T	T	T
F	F	F	T	F

We have

$$\begin{aligned} T_f &= \{(T,T,T),(T,F,F),(F,T,F),(F,F,T)\}, \\ F_f &= \{(T,T,F),(T,F,T),(F,T,T),(F,F,F)\}. \end{aligned}$$

For DNF of f, consider T_f. We apply the rule described on all four elements of T_f to get the associated blocks.

$$\begin{aligned} (T,T,T) &\implies (a \wedge b \wedge c), \\ (T,F,F) &\implies (a \wedge \overline{b} \wedge \overline{c}), \\ (F,T,F) &\implies (\overline{a} \wedge b \wedge \overline{c}), \\ (F,F,T) &\implies (\overline{a} \wedge \overline{b} \wedge c). \end{aligned}$$

Thus, the DNF of f is

$$(a \wedge b \wedge c) \vee (a \wedge \overline{b} \wedge \overline{c}) \vee (\overline{a} \wedge b \wedge \overline{c}) \vee (\overline{a} \wedge \overline{b} \wedge c).$$

In a similar manner, we use F_f of f to find CNF.

$$\begin{aligned} (T,T,F) &\implies (\overline{a} \vee \overline{b} \vee c), \\ (T,F,T) &\implies (\overline{a} \vee b \vee \overline{c}), \\ (F,T,T) &\implies (a \vee \overline{b} \vee \overline{c}), \\ (F,F,F) &\implies (a \vee b \vee c). \end{aligned}$$

Thus, the CNF of f is

$$(\overline{a} \vee \overline{b} \vee c) \wedge (\overline{a} \vee b \vee \overline{c}) \wedge (a \vee \overline{b} \vee \overline{c}) \wedge (a \vee b \vee c).$$

□

2.4 Problem

Problem 1: Define sets

$$A = \{1, \ldots, 10\}, B = \{3, 7, 11, 12\}, C = \{0, 1, \ldots, 20\}.$$

Which of the following are propositions?

(1) $1 + 1 = 3$
(2) $(A \cup B) \subseteq C$
(3) $A \cap B$
(4) $(8 + 22)^3/10^2$
(5) $7 \in A$
(6) $(B \cap C) \in 9$
(7) C is an infinite set

Problem 2: Find the possible wff's f and g in the following truth table.

a	b	f	g
T	T	F	T
T	F	T	T
F	T	T	F
F	F	T	T

Problem 3: If $p \to q$ is false, what is the truth value of

$$((\neg p) \wedge q) \longleftrightarrow (p \vee q)?$$

Problem 4: Construct the truth tables for the following:

1. $(a \longrightarrow T) \wedge (F \longrightarrow b)$
2. $(F \vee a) \longrightarrow (b \wedge F)$
3. $(a \vee b) \wedge (a \vee \neg b)$

Problem 5: Which of the following is a tautology?

1. $(a \longleftrightarrow b) \to (a \wedge b)$,
2. $(a \longleftrightarrow b) \longleftrightarrow (a \wedge b) \vee (\neg a \wedge \neg b)$

Problem 6: Show that

$$(a \vee b \longrightarrow c) \Longrightarrow (a \wedge b \longrightarrow c),$$

but the converse (i.e., $(a \wedge b \to c) \Rightarrow (a \vee b \to c)$) is not true.

Problem 7: Let p, q, r denote the following statements about a triangle ABC.

p : Triangle ABC is isosceles;
q : Triangle ABC is equilateral;
r : Triangle ABC is equiangular.

Translate each of the following into an English sentence.

1.	$q \longrightarrow p$	2.	$\neg p \longrightarrow \neg q$
3.	$q \longleftrightarrow r$	4.	$p \wedge \neg q$
5.	$r \longrightarrow p$		

Problem 8: Let p, q, r denote primitive statements. Use truth tables to prove the following logical equivalences.

1. $p \to (q \wedge r) \iff (p \to q) \wedge (p \to r)$
2. $[(p \vee q) \to r] \iff [(p \to r) \wedge (q \to r)]$

Problem 9: Let p, q, r denote primitive statements. Use the laws of logic to show that

$$[p \longrightarrow (q \vee r)] \iff [(p \wedge \neg q) \longrightarrow r].$$

Problem 10: Let p, q, and r be primitive statements. Write the dual for the following statements.

1. $q \longrightarrow p$
2. $p \longrightarrow (q \wedge r)$
3. $p \longleftrightarrow q$

Problem 11: Show that

$$((a \wedge b) \longrightarrow c) \iff ((a \longrightarrow c) \vee (b \longrightarrow c)).$$

Problem 12: Work out the truth tables for *modus ponens* to show that it is a logical implication but not an equivalence.

Problem 13: Consider

Premises: If there was a ball game, then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time.

Conclusion: There was no ball game.

Determine whether the conclusion follows logically from the premises. Explain by representing the statements symbolically and using rules of inference.

Problem 14: Consider

Premises: If Claghorn has wide support, then he'll be asked to run for the senate. If Claghorn yells "Eureka" in Iowa, he will not be asked to run for the senate. Claghorn yells "Eureka" in Iowa.

Conclusion: Claghorn does not have wide support.

Determine whether the conclusion follows logically from the premises. Explain by representing the statements symbolically and using rules of inference.

Problem 15: Write the converse, inverse, contrapositive, and negation of the following statement.

"If Sandra finishes her work, she will go to the basketball game."

Problem 16: Simplify

$$(p \wedge (\neg r \vee q \vee \neg q)) \vee ((r \vee t \vee \neg r) \wedge \neg q).$$

Problem 17: Simplify

$$(p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)) \wedge ((p \wedge r \wedge t) \vee t).$$

Problem 18: The following is a logical proof for

$$(p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)) \rightarrow (s \vee t).$$

Refer to the laws of logic and inference rule, and give reasons to justify each step of the proof.

steps		reasons
1.	p	
2.	$p \rightarrow q$	
3.	q	
4.	$r \rightarrow \neg q$	
5.	$q \rightarrow \neg r$	
6.	$\neg r$	
7.	$s \vee r$	
8.	s	
9.	$s \vee t$	

Problem 19: The following is a logical proof for the inference:

Premises :	$(\neg p \vee q) \rightarrow r$
	$r \rightarrow (s \vee t)$
	$\neg s \wedge \neg u$
	$\neg u \rightarrow \neg t$
Conclusion :	p

Give a reason to justify each step of the proof.

	steps	reasons
1.	$\neg s \wedge \neg u$	
2.	$\neg u$	
3.	$\neg u \rightarrow \neg t$	
4.	$\neg t$	
5.	$\neg s$	
6.	$\neg s \wedge \neg t$	
7.	$r \rightarrow (s \vee t)$	
8.	$\neg(s \vee t) \rightarrow \neg r$	
9.	$(\neg s \wedge \neg t) \rightarrow \neg r$	
10.	$\neg t$	
11.	$(\neg p \vee q) \rightarrow r$	
12.	$\neg r \rightarrow \neg(\neg p \vee q)$	
13.	$\neg r \rightarrow (p \vee \neg q)$	
14.	$p \wedge \neg q$	
15.	p	

Problem 20: The following is a logical proof for

$$((p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)) \rightarrow (\neg q \rightarrow s).$$

	steps	reasons
1.	$\neg(\neg q \rightarrow s)$	
2.	$\neg q \wedge \neg s$	
3.	$\neg s$	
4.	$\neg r \vee s$	
5.	$\neg r$	
6.	$p \rightarrow q$	
7.	$\neg q$	
8.	$\neg p$	
9.	$p \vee r$	
10.	r	
11.	$\neg r \wedge r$	
12.	$\neg q \rightarrow s$	

1. Give a reason to justify each step of the proof. (Note: This is a proof by contradiction.)
2. Give a direct proof.

Problem 21: Prove that the following inference is valid.

$$\begin{array}{rl} \text{Premises}: & \neg p \leftrightarrow q \\ & q \rightarrow r \\ & \neg r \\ \hline \text{Conclusion}: & p \end{array}$$

Problem 22: Show that the following premises are inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books.

Problem 23: Let $D_x := \mathbf{R}$. Which of the following are predicates?

$$\begin{array}{ll} (1) & x^2+1<0 \\ (2) & x \text{ is odd} \\ (3) & (x^2-1)/(x+1) \\ (4) & 1+2=3 \\ (5) & x \in \mathbf{N} \\ (6) & \sin^2 x + \cos^2 x \end{array}$$

Problem 24: Define

$$\begin{aligned} A &= \{x | x \in \mathbf{N},\ x \leq 10\}; \\ B &= \{y;\ y \in \mathbf{N},\ y \leq 15,\ y \text{ is even}\}. \end{aligned}$$

Write two predicates, of which $A - B$ and $B - A$ are the truth sets respectively.

Problem 25: Let P and Q be two predicates, and let their truth sets be denoted as T_P, T_Q, and falsity sets be denoted as F_P, F_Q. Prove the following identities.

1. $T_P \cap T_Q = T_{P \wedge Q}$
2. $T_P \cup T_Q = T_{P \vee Q}$
3. $F_P \cap F_Q = F_{P \vee Q}$
4. $F_P \cup F_Q = F_{P \wedge Q}$

Problem 26: What are the most general conditions on the truth sets T_P and T_Q making

1. $(P(x) \longrightarrow Q(x)) \Longrightarrow P(x)$ true
2. $(P(x) \longrightarrow Q(x)) \Longrightarrow Q(x)$ true

Problem 27: Suppose the domain D for predicates P, Q, and S is $\{a, b, c\}$. Express the following propositions without using quantifiers.

1. $\forall x\ P(x)$
2. $(\forall x\ R(x)) \wedge (\exists x\ S(x))$

Problem 28: Let $D_x = D_y = \{1, 2, 3, 4, 5\}$. Define the predicate $P(x, y)$ as

$$P(x, y) \ := \ (y \geq x) \text{ or } (x + y > 6).$$

Find the truth sets of the following predicates:

1. $P(x, y)$.
2. $\exists x P(x, y)$
3. $\exists y P(x, y)$
4. $\forall x P(x, y)$
5. $\forall y P(x, y)$

Problem 29: Let V be the truth set of $P(x, y)$. Thus, $V \subseteq D_x \times D_y$.

1. Prove that the truth set of $\exists x \in D_x P(x, y)$ is the set of all second coordinates of ordered pairs in V.
2. Prove that the truth set of $\forall x \in D_x P(x, y)$ is

$$\{b;\ b \in D_y, \quad D_x \times \{b\} \subseteq V\}.$$

Problem 30: Let sets

$$\begin{aligned} D_x &= \{t;\ t \in \mathbf{R},\ -1 \leq t \leq 1\} \text{ and} \\ D_y &= \{r;\ r \in \mathbf{R},\ 0 \leq r \leq 1\} \end{aligned}$$

be the universes of x and y, respectively. Define

$$\begin{aligned} P(x, y) &= (x + y \leq 1) \wedge (y - x \leq 1); \\ Q(x, y) &= x^2 + y^2 \leq 1. \end{aligned}$$

Prove that $T_P \subseteq T_Q$, where T_P is the truth set of P and T_Q is the truth set of Q.

Problem 31: Let A, B and S be sets. The following is a wrong proof to claim that

$$S \subseteq (A \cup B) \Rightarrow (S \subseteq A \text{ or } S \subseteq B).$$

Wrong proof: If $S \subseteq (A \cup B)$, then

$$\begin{aligned} & x \in S \Rightarrow x \in (A \cup B), && (1) \\ & x \in S \Rightarrow (x \in A \text{ or } x \in B), && (2) \\ & (x \in S \Rightarrow x \in A) \text{ or } (x \in S \Rightarrow x \in B), && (3) \\ & S \subseteq A \text{ or } S \subseteq B. && (4) \end{aligned}$$

Point out which step is problematic and explain why.

[Hint: consider the definition of subset in term of quantified predicates.]

Problem 32: Let $P(x, y)$ be a predicate defined as

$$P(x, y) : (x \vee y) \rightarrow z.$$

Express the negation of $\forall x \exists y \ P(x, y)$ without "$\neg$" in front of any quantifier.

Problem 33: Find the negations of the following two quantified predicates without "$\neg$" in front of any quantifier.

1. $\forall x \ \forall y \ [(x > y) \rightarrow (x - y > 0)]$.
2. $\forall x \ \forall y \ [(x < y) \rightarrow \exists z(x < z < y)]$.

Problem 34: Take $P1$ through $P7$ as premises. See what conclusion you can logically derive. Explain.

$P1$: All the policemen on this beat eat with our cook.

$P2$: No man with long hair can fail to be a poet.

$P3$: Amos Judd has never been in prison.

$P4$: Our cook's cousins all love cold mutton.

$P5$: None but policemen on this beat are poets.

$P6$: None but her cousins ever eat with the cook.

$P7$: Men with short hair have all been to prison.

Problem 35: Consider "No pigs have wings". Write this proposition as a quantified predicate.

Problem 36: Consider

Premises:
All soldiers can march.
Some babies are not soldiers.

Conclusion:
Some babies cannot march.

Determine whether the conclusion follows logically from the premises. Explain.

Problem 37: Let $D_x = \mathbf{N}$ and $D_y = \mathbf{N}^0$. Define $P(x, y)$ as "x divides y". Find the truth values of the following quantified predicates.

1. $\forall y P(1, y)$
2. $\forall x P(x, 0)$
3. $\forall x P(x, x)$
4. $\forall y \exists x P(x, y)$
5. $\exists y \forall x P(x, y)$

6. $\forall x \forall y[(P(x,y) \wedge P(y,x)) \rightarrow (x = y)]$
7. $\forall x \forall y \forall x[(P(x,y) \wedge P(y,x)) \rightarrow P(x,z)]$

Problem 38: Let D_x and D_y denote the domains of x and y, respectively. Consider the following quantified statement

$$\forall x \; \exists y \; [x + y = 17].$$

Determine the truth value of the quantified predicate in different domains.

1. $D_x = D_y =$ the set of integers.
2. $D_x = D_y =$ the set of positive integers.
3. $D_x =$ the set of integers and $D_y =$ the set of positive integers.
4. $D_x =$ the set of positive integers and $D_y =$ the set of integers.

Problem 39: What is the DNF of

$$(a \longrightarrow b) \wedge (\neg a \longrightarrow \neg b)?$$

Problem 40: What is the DNF of

$$(a \longrightarrow b) \wedge (a \longrightarrow \neg b)?$$

Problem 41: Find the CNF's of

1. $a \longrightarrow \neg b$.
2. $(a \wedge b) \vee c$.

Problem 42: Let the wff f be $a \wedge (b \longleftrightarrow c)$.

1. Use the shortcut method (use the falsity set) to find the CNF of f.
2. Use the propositional calculus to find the DNF of f.

Problem 43: Consider the following mathematical statement in number theory:

"For every integer n bigger than 1, there is a prime strictly between n and $2n$."

1. Express the statement in terms of quantifiers, variable(s), predicates, and the inequality symbols $<$ or $>$.
2. Express the negation of the predicate found in 1 without using $\neg$.

[Be careful to define the domain(s) of your variable(s)]

Problem 44: Let x and y range over all integers. Prove that for all x, y, if xy is even, then at least one of x and y is even.

Problem 45: Are the following arguments logically correct?

Premises:
All who are anxious to learn work hard.
Some of these boys work hard.

Conclusion:
Therefore some of these boys are anxious to learn.

Problem 46: Are the following arguments logically correct?

Premises:
There are men who are soldiers.
All soldiers are strong.
All soldiers are brave.

Conclusion:
Therefore some strong men are brave.

Problem 47: Let the universe be a social club, and let x and y range over the members of the club. Define the predicate $P(x, y)$ as

$$P(x, y) := x \text{ loves } y.$$

Translate the following quantified predicates into English sentences

1. $\forall x \forall y \ P(x, y)$
2. $\exists x \exists y \ P(x, y)$
3. $\forall x \exists y \ P(x, y)$
4. $\exists x \forall y \ P(x, y)$

Problem 48: Let the domain range over all real numbers. Find a possible conclusion from the given premises.

Premises:
All integers are rational numbers.
The real number π is not a rational number.

Problem 49: Let the domain range over all people in the USA. Find a possible premise for the following inference.

Premises:
All librarians know the Library of Congress Classification System.
(unknown premise)

Conclusion:
Margaret knows the Library of Congress Classification System.

Problem 50: Let P and Q be two predicates. Prove that

$$\exists x[P(x) \vee Q(x)] \iff \exists x P(x) \vee \exists x Q(x).$$

Problem 51: Let P and Q be two predicates. Prove that

$$\forall x[P(x) \wedge Q(x)] \iff \forall x P(x) \wedge \forall x Q(x).$$

Problem 52: Let P and Q be two predicates. Prove that

$$\forall x P(x) \vee \forall x Q(x) \quad \Rightarrow \quad \forall x[P(x) \vee Q(x)].$$

Problem 53: Let P and Q be two predicates. Disprove that

$$\forall x[P(x) \vee Q(x)] \quad \Rightarrow \quad \forall x P(x) \vee \forall x Q(x).$$

Problem 54: Prove or disprove the following statement.

There exist a and b, where a and b are irrational and a^b is rational.

[Hint: Use the fact that $\sqrt{2}$ is irrational; a and b don't have to be distinct.]

2.5 Solutions

Solution 1: $(1), (2), (5)$, and (7) are propositions.

$(3), (4)$ and (6) are not propositions.

Note: (6) $(B \cap C) \in 9$ is not a proposition. It is meaningless, but that does not mean its truth value is F.

$\square$

Solution 2: There are infinitely many wff's that satisfy the given truth table. The simplest two we can think of are

$$f = \neg(a \wedge b), \text{ and } \quad g = b \longrightarrow a.$$

$\square$

Solution 3: Let $p \rightarrow q$ be false. The only case in which $p \rightarrow q$ is false is when $p = T$ and $q = F$. We can replace the occurrences of p and q by their values and find the result step by step as the following:

$$\begin{aligned} & ((\neg p) \wedge q) \leftrightarrow (p \vee q) \\ = \; & ((\neg T) \wedge F) \leftrightarrow (T \vee F) \\ = \; & (F \wedge F) \leftrightarrow T \\ = \; & F \leftrightarrow T \\ = \; & F \end{aligned}$$

Refer to the laws of logic to justify each step above. $\square$

Solution 4:

1. Let $f(a,b) = (a \to T) \wedge (F \to b)$

a	b	T	F	$a \to T$	$F \to b$	$f(a,b)$
T	T	T	F	T	T	T
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	T	F	T	T	T

2. Let $f(a,b) = (F \vee a) \to (b \wedge F)$

a	b	T	F	$F \vee a$	$b \wedge F$	$f(a,b)$
T	T	T	F	T	F	F
T	F	T	F	T	F	F
F	T	T	F	F	F	T
F	F	T	F	F	F	T

3. Let $f(a,b) = (a \vee b) \wedge (a \vee \neg b)$.

a	b	$\neg b$	$a \vee b$	$a \vee \neg b$	$f(a,b)$
T	T	F	T	T	T
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

□

Solution 5: We will check the truth tables to tell whether the statements are tautologies or not.

1. Let $f = (a \leftrightarrow b) \to (a \wedge b)$.

a	b	$a \leftrightarrow b$	$a \wedge b$	f
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	F

The last column of the truth table contains an F in the last row. Thus, f is not a tautology.

2. Let $g = (a \leftrightarrow b) \leftrightarrow (a \wedge b) \vee (\neg a \wedge \neg b)$.

Note: g should be read as $(a \leftrightarrow b) \leftrightarrow ((a \wedge b) \vee (\neg a \wedge \neg b))$, but not as $((a \leftrightarrow b) \leftrightarrow (a \wedge b)) \vee (\neg a \wedge \neg b)$, because the precedence priority of $\vee$ is higher than the precedence priority of $\leftrightarrow$.

$$s = (a \wedge b) \vee (\neg a \wedge \neg b).$$

a	b	$a \leftrightarrow b$	$a \wedge b$	$\neg a \wedge \neg b$	s	g
T	T	T	T	F	T	T
T	F	F	F	F	F	T
F	T	F	F	F	F	T
F	F	T	F	T	T	T

Therefore, g is a tautology because every truth value in the last column of its truth table is true.

□

Solution 6: Let

$$\begin{aligned} p &= a \vee b \longrightarrow c \\ q &= a \wedge b \longrightarrow c \\ r &= (a \vee b \longrightarrow c) \longrightarrow (a \wedge b \longrightarrow c) \\ s &= (a \vee b \longrightarrow c) \longleftarrow (a \wedge b \longrightarrow c) \end{aligned}$$

a	b	c	$a \wedge b$	$a \vee b$	p	q	r	s
T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	T	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	T	T	F
F	T	T	F	T	T	T	T	T
F	T	F	F	T	F	T	T	F
F	F	T	F	F	T	T	T	T
F	F	F	F	F	T	T	T	T

From the truth table above, r is a tautology and s is not. Therefore,

$$(a \vee b \longrightarrow c) \Longrightarrow (a \wedge b \longrightarrow c),$$

but its converse is not true.

□

Solution 7: Let p: Triangle ABC is isosceles;
q: Triangle ABC is equilateral;
r: Triangle ABC is equiangular.

1. $q \longrightarrow p$: If triangle ABC is equilateral, then it is isosceles.
2. $\neg p \longrightarrow \neg q$: If triangle ABC is not isosceles, then it is not equilateral.
3. $q \longleftrightarrow r$: Triangle ABC is equilateral if and only if it is equiangular.
4. $p \wedge \neg q$: Triangle ABC is isosceles, but not equilateral.
5. $r \longrightarrow p$: If triangle ABC is equiangular, then it is isosceles.

$\square$

Solution 8:

1. To prove that $p \to (q \wedge r) \iff (p \to q) \wedge (p \to r)$, let

$$\begin{aligned} s &= p \to (q \wedge r), \\ t &= (p \to q) \wedge (p \to r), \\ u &= (p \to (q \wedge r)) \leftrightarrow ((p \to q) \wedge (p \to r)). \end{aligned}$$

p	q	r	$q \wedge r$	$p \to q$	$p \to r$	s	t	u
T	T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F	T
T	F	T	F	F	T	F	F	T
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Since the last column contains truth value T for all possible values of $p.q$, and r, therefore, $p \to (q \wedge r)$ and $(p \to q) \wedge (p \to r)$ are logically equivalent. $\square$

2. To prove that $((p \vee q) \to r) \iff ((p \to r) \wedge (q \to r))$, let

$$\begin{aligned} s &= (p \vee q) \to r, \\ t &= (p \to r) \wedge (q \to r), \\ u &= ((p \vee q) \to r) \leftrightarrow ((p \to r) \wedge (q \to r)), \end{aligned}$$

and construct the truth table:

p	q	r	$p \vee q$	$p \to r$	$q \to r$	s	t	u
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Clearly, $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$ are logically equivalent.

□

Solution 9: Using the laws of logic we obtain:

$$\begin{array}{lr}
p \to (q \vee r) & \\
\iff \neg p \vee (q \vee r) & \text{Logical equivalence} \\
\iff (\neg p \vee q) \vee r & \text{Associative law} \\
\iff \neg\neg(\neg p \vee q) \vee r & \text{Double negation law} \\
\iff \neg(\neg\neg p \wedge \neg q) \vee r & \text{De Morgan's law} \\
\iff \neg(p \wedge \neg q) \vee r & \text{Double negation law} \\
\iff (p \wedge \neg q) \to r & \text{Logical equivalence}
\end{array}$$

Therefore, $[p \longrightarrow (q \vee r)] \iff [(p \wedge \neg q) \longrightarrow r]$.

□

Solution 10: From the definition of duality it is not possible to give the dual of a logical statement that contains "$\to$" or "$\leftrightarrow$". We have to find its logical equivalent statements that contain no logical connectives other than "$\wedge$" and "$\vee$".

1. Since $q \to p \equiv \neg q \vee p$, hence the dual of $q \to p$ is $\neg q \wedge p$.

2. Since $p \to (q \wedge r) \equiv \neg p \vee (q \wedge r)$ its dual is $\neg p \wedge (q \vee r)$.

3. Reduction of $p \leftrightarrow q$ to a formula that contains connectives only $\wedge, \vee$, and

$\neg$ is given below.

$$\begin{aligned} p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) \\ &\equiv (\neg p \vee q) \wedge (\neg q \vee p) \\ &\equiv [(\neg p \vee q) \wedge \neg q] \vee [(\neg p \vee q) \wedge p] \\ &\equiv (\neg p \wedge \neg q) \vee (q \wedge \neg q) \vee (\neg p \wedge p) \vee (q \wedge p) \\ &\equiv (\neg p \wedge \neg q) \vee F \vee F \vee (q \wedge p) \\ &\equiv (\neg p \wedge \neg q) \vee (q \wedge p). \end{aligned}$$

Thus, the dual of $p \leftrightarrow q$ is $(\neg p \vee \neg q) \wedge (q \vee p)$.

□

Solution 11: Our goal is to show that the values in the last column of the truth table are all true. Let,

$$\begin{aligned} &s = (a \wedge b) \to c, \\ &t = (a \to c) \vee (b \to c), \\ &u = ((a \wedge b) \to c) \leftrightarrow ((a \to c) \vee (b \to c)). \end{aligned}$$

a	b	c	$a \wedge b$	$a \to c$	$b \to c$	s	t	u
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	F	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Therefore, u is a tautology. □

Solution 12: Recall that the Modus Ponens rule is: $(a \wedge (a \to b)) \Rightarrow b$. Let,

$$\begin{aligned} &s = (a \wedge (a \to b)) \to b, \\ &t = (a \wedge (a \to b)) \leftrightarrow b. \end{aligned}$$

a	b	$a \to b$	$a \wedge (a \to b)$	s	t
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	F	T	F
F	F	T	F	T	T

From the truth table above, we know that s is a tautology. Therefore,

$$(a \wedge (a \to b)) \Rightarrow b.$$

But t is not a tautology, and hence the Modus Ponens is not a logical equivalence. □

Solution 13: Let BG: there was a ball game,
TD: traveling was difficult,
AO: they arrived on time.

Premises: $BG \longrightarrow TD, \quad AO \longrightarrow \neg TD, \quad AO.$

Conclusion: $\neg BG.$

	steps	*reasons*
1.	$BG \to TD$	Premises
2.	$AO \to \neg TD$	Premises
3.	AO	Premises
4.	$\neg TD$	2,3, Modus Ponens
5.	$\neg TD \to \neg BG$	1, Contrapositive
6.	$\neg BG$	4,5, Modus Ponens

Therefore, the conclusion that there was no ball game is logically correct based on the given premises. □

Solution 14: Let CS: Claghorn has wide support,
RS: Claghorn is asked to run for the senate,
CY: Claghorn yells "Eureka"

Premises: $CS \longrightarrow RS, \quad CY \longrightarrow \neg RS, \quad CY.$

Conclusion: $\neg CS.$

	steps	*reasons*
1.	$CS \to RS$	Premises
2.	$CY \to \neg RS$	Premises
3.	CY	premises
4.	$CY \wedge (CY \to \neg RS)$	$2 \wedge 3$
5.	$\neg RS$	4, Modus Ponens
6.	$\neg RS \to \neg CS$	1, Contrapositive
7.	$\neg RS \wedge (\neg RS \to \neg CS)$	$5 \wedge 6$
8.	$\neg CS$	7, Modus Ponens

Therefore, "Claghorn doesn't have wide support" is logically correct from the given premises. □

Solution 15: Let p : Sandra finishes her work.
q : Sandra goes to the basketball game.

Implication: $(p \to q)$
If Sandra finishes her work, she will go to the basketball game.

Converse: $(q \to p)$
If Sandra goes to the basketball game, she will finish her work.

Inverse: $(\neg p \to \neg q)$
If Sandra does not finish her work, she will not go to the basketball game.

Contrapositive: $(\neg q \to \neg p)$
If Sandra does not go to the basketball game, she does not finish her work.

Negation: $(p \wedge \neg q)$
Sandra finishes her work, and she does not go to the basketball game.

□

Solution 16:

$$\begin{aligned} &(p \wedge (\neg r \vee q \vee \neg q)) \vee ((r \vee t \vee \neg r) \wedge \neg q) \\ &\quad \Leftrightarrow \quad (p \wedge (\neg r \vee T)) \vee ((t \vee T) \wedge \neg q) \\ &\quad \Leftrightarrow \quad (p \wedge T) \vee (T \wedge \neg q) \\ &\quad \Leftrightarrow \quad p \vee \neg q. \end{aligned}$$

□

Solution 17:

$(p \vee (p \wedge q) \vee (p \wedge q \wedge \neg r)) \wedge ((p \wedge r \wedge t) \vee t)$

$$
\begin{aligned}
&\Leftrightarrow (p \vee (p \wedge q \wedge T) \vee (p \wedge q \wedge \neg r)) \wedge ((p \wedge r \wedge t) \vee (T \wedge t) && (1)\\
&\Leftrightarrow (p \vee (((p \wedge q) \wedge T) \vee ((p \wedge q) \wedge \neg r))) \wedge (((p \wedge r) \wedge t) \vee (T \wedge t) && (2)\\
&\Leftrightarrow (p \vee ((p \wedge q) \wedge (T \vee \neg r)) \wedge (((p \wedge r) \vee T) \wedge t) && (3)\\
&\Leftrightarrow (p \vee ((p \wedge q) \wedge T) \wedge (T \wedge t) && (4)\\
&\Leftrightarrow (p \vee (p \wedge q)) \wedge t) && (5)\\
&\Leftrightarrow ((p \wedge T) \vee (p \wedge q)) \wedge t && (6)\\
&\Leftrightarrow (p \wedge (T \vee q)) \wedge t && (7)\\
&\Leftrightarrow (p \wedge T) \wedge t && (8)\\
&\Leftrightarrow p \wedge t && (9)
\end{aligned}
$$

Explanation: In (1) identity law is used to have $(p \wedge q) \Leftrightarrow (p \wedge q \wedge T)$ and $t \Leftrightarrow (T \wedge t)$. Associative law is use is (2). Distributive law is used in (3). Domination law is used is (4). Identity law is used in (5) and (6) to drop two T's and add one T, respectively. Distributive law is used in (7). Domination law is used is (8). Finally, we use identity law in the last step. □

Solution 18: To prove $(p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)) \rightarrow (s \vee t)$, we will take p, $p \rightarrow q$, $s \vee r$, and $r \rightarrow \neg q$ as the given assumptions.

	steps	*reasons*
1.	p	Assumption
2.	$p \rightarrow q$	Assumption
3.	q	1,2, Modus Ponens
4.	$r \rightarrow \neg q$	Assumption
5.	$q \rightarrow \neg r$	4, Contrapositive
6.	$\neg r$	3,5, Modus Ponens
7.	$s \vee r$	Assumption
8.	s	6,7, Disjunctive Syllogism
9.	$s \vee t$	8, Disjunctive Amplification

□

Solution 19:

Premises :	$(\neg p \vee q) \rightarrow r,\ r \rightarrow (s \vee t),\ \neg s \wedge \neg u,\ \neg u \rightarrow \neg t$
Conclusion :	p

	steps	*reasons*
1.	$\neg s \wedge \neg u$	Premises
2.	$\neg u$	1, Conjunctive Simplification
3.	$\neg u \to \neg t$	Premises
4.	$\neg t$	2,3, Modus Ponens
5.	$\neg s$	1, Conjunctive Simplification
6.	$\neg s \wedge \neg t$	4,5, Conjunction
7.	$r \to (s \vee t)$	Premises
8.	$\neg(s \vee t) \to \neg r$	7, Contrapositive
9.	$(\neg s \wedge \neg t) \to \neg r$	8, De Morgan's law
10.	$\neg r$	6,9, Modus Ponens
11.	$(\neg p \vee q) \to r$	Premises
12.	$\neg r \to \neg(\neg p \vee q)$	11, Contrapositive
13.	$\neg r \to (p \vee \neg q)$	12, De Morgan's law
14.	$p \wedge \neg q$	10,13, Modus Ponens
15.	p	14, Conjunctive Simplification

□

Solution 20:

$$((p \to q) \wedge (\neg r \vee s) \wedge (p \vee r)) \to (\neg q \to s) \tag{2.4}$$

1. The following proof for (4) is a contradiction argument. By contradiction, we assume that the premises and the negation of the consequence are both true, i.e., we have

   Assumptions: $p \to q,\ \neg r \vee s,\ p \vee r,\ \neg(\neg q \to s)$.

	steps	*reasons*
1.	$\neg(\neg q \to s)$	Assumption, by Contradiction)
2.	$\neg q \wedge \neg s$	Negation of implication
3.	$\neg s$	2, Conjunctive Simplification
4.	$\neg r \vee s$	Assumption
5.	$\neg r$	3,4, Disjunctive syllogism
6.	$p \to q$	Assumption
7.	$\neg q$	2, Conjunctive Simplification
8.	$\neg p$	6,7, Modus Tollens
9.	$p \vee r$	Assumption
10.	r	8,9, Disjunctive syllogism
11.	$\neg r \wedge r$	5,10, Conjunction
12.	$\neg q \to s$	Since 11 is a contradiction, 1 can't be true

□

2. A direct proof for (4).

   Assumptions: $p \to q,\ \neg r \vee s,\ p \vee r$.

steps		reasons
1.	$p \to q$	Assumption
2.	$\neg q \to \neg p$	Equivalence of 1
3.	$p \vee r$	Assumption
4.	$\neg p \to r$	Equivalence of 3
5.	$\neg q \to r$	2,4, Syllogism
6.	$\neg r \vee s$	Assumption
7.	$r \to s$	Equivalence of 6
8.	$\neg q \to s$	5,7, Syllogism

□

Solution 21:

$$\begin{array}{rl} \text{Premises :} & \neg p \leftrightarrow q,\ q \to r,\ \neg r \\ \hline \text{Conclusion :} & p \end{array}$$

steps		*reasons*
1.	$\neg p \leftrightarrow q$	Premises
2.	$\neg p \to q$	From 1
3.	$q \to \neg p$	From 1
4.	$q \to r$	Premises
5.	$\neg r \to \neg q$	4, Contrapositive
6.	$\neg r$	Premises
7.	$\neg q$	5,6 Modus Ponens
8.	$\neg q \to p$	2 Contrapositive
9.	p	7,8 Modus Ponens

□

Solution 22: We say a system (a set of premises) is inconsistent if and only if we can obtain some results from the system that contradict each other.

Let MC: Jack misses many classes.
FS: Jack fails school.
UE: Jack is uneducated.
RB: Jack reads a lot of books.

$$\text{Premises :} \quad MC \to FS,\ FS \to UE,\ RB \to \neg EU,\ MC \wedge RB.$$

	steps	*reasons*
1.	$MC \wedge RB$	Premises
2.	MC	1, Conjunctive Simplification
3.	$MC \longrightarrow FS$	Premises
4.	FS	2,3, Modus Ponens
5.	$FS \longrightarrow UE$	Premises
6.	UE	4,5, Modus Ponens
7.	$RB \longrightarrow \neg UE$	Premises
8.	RB	1, Conjunctive Simplification
9.	$\neg UE$	7,8, Modus Ponens
10.	$UE \wedge \neg UE$	6,9, Conjunction

We have both UE and $\neg UE$, and hence the premises are inconsistent. □

Solution 23: Recall the definition of predicates: A statement is a predicate if we can replace every variable in the statement by any instance in its domain to form a proposition.

(1), (2), (4) and (5) are predicates. (3) and (6) are not predicates.

Note: A predicate may not have any variable. Therefore, all propositions are also predicates, and hence (4) is a predicate.

□

Solution 24: Let $D_x = \mathbf{N}$ be the universe of x. We find $A - B$ and $B - A$ as

$$A - B = \{1, 3, 5, 7, 9\} \text{ and } B - A = \{12, 14\}.$$

Define predicates $P(x)$ and $Q(x)$ as

$$P(x) = (x \in A) \wedge (x \notin B) \text{ and } Q(x) = (x \notin A) \wedge (x \in B).$$

Thus, $T_P = A - B$ and $T_Q = B - A$.

Note: One does not give, for example, $\{x : x \in \mathbf{N}, 10 < x \leq 15, x \text{ is even }\}$ as the answer because it is the true-set of Q, but not a predicate with the truth set $B - A$.

□

Solution 25:

1. $T_P \cap T_Q = T_{P\wedge Q}$

$$\begin{aligned} x \in T_P \cap T_Q &\Leftrightarrow x \in T_P \text{ and } x \in T_Q \\ &\Leftrightarrow P(x) = T \text{ and } Q(x) = T \\ &\Leftrightarrow (P \wedge Q)(x) = T \\ &\Leftrightarrow x \in T_{P\wedge Q}. \end{aligned}$$

2. $T_P \cup T_Q = T_{P\vee Q}$

$$\begin{aligned} x \in T_P \cup T_Q &\Leftrightarrow x \in T_P \text{ or } x \in T_Q \\ &\Leftrightarrow P(x) = T \text{ or } Q(x) = T \\ &\Leftrightarrow (P \vee Q)(x) = T \\ &\Leftrightarrow x \in T_{P\vee Q}. \end{aligned}$$

3. $F(P) \cap F(Q) = F_{P\vee Q}$

$$\begin{aligned} x \in F(P) \cap F(Q) &\Leftrightarrow x \in F(P) \text{ and } x \in F(Q) \\ &\Leftrightarrow P(x) = F \text{ and } Q(x) = F \\ &\Leftrightarrow (P \vee Q)(x) = F \qquad \text{(why?)} \\ &\Leftrightarrow x \in F_{P\vee Q}. \end{aligned}$$

4. $F(P) \cup F(Q) = F_{P\wedge Q}$

$$\begin{aligned} x \in F(P) \cup F(Q) &\Leftrightarrow x \in F(P) \text{ or } x \in F(Q) \\ &\Leftrightarrow P(x) = F \text{ or } Q(x) = F \\ &\Leftrightarrow (P \wedge Q)(x) = F \qquad \text{(why?)} \\ &\Leftrightarrow x \in F_{P\wedge Q}. \end{aligned}$$

□

Solution 26: Let $T_{P\to Q}$ denote the truth set of $P(x) \to Q(x)$.

1. $(P(x) \to Q(x)) \Rightarrow P(x)$.

$$\begin{aligned} [(P(x) \to Q(x)) \Rightarrow P(x)] &\Leftrightarrow T_{P\to Q} \subseteq T_P \\ &\Leftrightarrow (F_P \cup T_Q) \subseteq T_P \\ &\Leftrightarrow (\overline{T_P} \cup T_Q) \subseteq T_P \\ &\Leftrightarrow \overline{T_P} \subseteq T_P \text{ and } T_Q \subseteq T_P. \end{aligned}$$

The only possibility to make $\overline{T_P} \subseteq T_P$ is when $\overline{T_P} = \emptyset$, namely, $T_P = U$. If $T_P = U$, then $T_Q \subseteq T_P$ is satisfied for any T_Q. Therefore, $T_P = U$ is the most general condition for $(P(x) \to Q(x)) \Rightarrow P(x)$ to be true.

2. $(P(x) \to Q(x)) \Rightarrow Q(x)$.

$$\begin{aligned}
[(P(x) \to Q(x)) \Rightarrow Q(x)] &\Leftrightarrow T_{P\to Q} \subseteq T_Q \\
&\Leftrightarrow (F_P \cup T_Q) \subseteq T_Q \\
&\Leftrightarrow F_P \subseteq T_Q \text{ and } T_Q \subseteq T_Q \\
&\Leftrightarrow F_P \subseteq T_Q.
\end{aligned}$$

Therefore, $F_P \subseteq T_Q$ is the most general condition for $(P(x) \to Q(x)) \Rightarrow Q(x)$ to be true.

Note: The most general condition means the least restricted or the weakest condition. For example, consider $1 < a < 10$ and $1 < a$. $1 < a$ is weaker than $1 < a < 10$, hence $1 < a$ is more general than $1 < a < 10$. □

Solution 27: Let $D_x = \{a, b, c\}$.

1. $\forall x P(x) = P(a) \wedge P(b) \wedge P(c)$.
2. $(\forall x R(x)) \wedge (\exists x S(x)) = (R(a) \wedge R(b) \wedge R(c)) \wedge (S(a) \vee S(b) \vee S(c))$.

□

Solution 28: Let $D_x = D_y = \{1, 2, 3, 4, 5\}$, and $P(x, y) := (y \geq x) \vee (x + y > 6)$. Thus, the truth set of $P(x, y)$ is a subset of $D_x \times D_y$.

1. $T_{P(x,y)} = T_{y \geq x} \cup T_{x+y>6}$.

$$T_{y\geq x} = \left\{ \begin{array}{l} (1,1), \\ (1,2), (2,2), \\ (1,3), (2,3), (3,3), \\ (1,4), (2,4), (3,4), (4,4), \\ (1,5), (2,5), (3,5), (4,5), (5,5) \end{array} \right\},$$

$$T_{x+y>6} = \left\{ \begin{array}{r} (5,2), \\ (4,3), (5,3), \\ (3,4), (4,4), (5,4), \\ (2,5), (3,5), (4,5), (5,5) \end{array} \right\}.$$

$$T_{y\geq x} \cup T_{x+y>6} = \left\{ \begin{array}{l} (1,1), \\ (1,2),(2,2), \qquad\qquad\qquad (5,2), \\ (1,3),(2,3),(3,3),(4,3),(5,3), \\ (1,4),(2,4),(3,4),(4,4),(5,4), \\ (1,5),(2,5),(3,5),(4,5),(5,5) \end{array} \right\}.$$

2. $T_{\exists x P(x,y)} = T_{P(1,y)} \cup T_{P(2,y)} \cup T_{P(3,y)} \cup T_{P(4,y)} \cup T_{P(5,y)}$.

$$\begin{aligned} T_{P(1,y)} &= \{y : (y \geq 1) \vee (y > 5)\} = \{1,2,3,4,5\}, \\ T_{P(2,y)} &= \{y : (y \geq 2) \vee (y > 4)\} = \{2,3,4,5\}, \\ T_{P(3,y)} &= \{y : (y \geq 3) \vee (y > 3)\} = \{3,4,5\}, \\ T_{P(4,y)} &= \{y : (y \geq 4) \vee (y > 2)\} = \{3,4,5\}, \\ T_{P(5,y)} &= \{y : (y \geq 5) \vee (y > 1)\} = \{2,3,4,5\}. \end{aligned}$$

Therefore, $T_{\exists x P(x,y)} = \{1,2,3,4,5\}$.

3. $T_{\exists y P(x,y)} = T_{P(x,1)} \cup T_{P(x,2)} \cup T_{P(x,3)} \cup T_{P(x,4)} \cup T_{P(x,5)}$.

$$\begin{aligned} T_{P(x,1)} &= \{x : (1 \geq x) \vee (x > 5)\} = \{1\}, \\ T_{P(x,2)} &= \{x : (2 \geq x) \vee (x > 4)\} = \{1,2,5\}, \\ T_{P(x,3)} &= \{x : (3 \geq x) \vee (x > 3)\} = \{1,2,3,4,5\}, \\ T_{P(x,4)} &= \{x : (4 \geq x) \vee (x > 2)\} = \{1,2,3,4,5\}, \\ T_{P(x,5)} &= \{x : (5 \geq x) \vee (x > 1)\} = \{2,3,4,5\}. \end{aligned}$$

Therefore, $T_{\exists y P(x,y)} = \{1,2,3,4,5\}$.

4. $T_{\forall x P(x,y)} = T_{P(1,y)} \cap T_{P(2,y)} \cap T_{P(3,y)} \cap T_{P(4,y)} \cap T_{P(5,y)} = \{3,4,5\}$.

5. $T_{\forall y P(x,y)} = T_{P(x,1)} \cap T_{P(x,2)} \cap T_{P(x,3)} \cap T_{P(x,4)} \cap T_{P(x,5)} = \{1\}$.

□

Solution 29: Let $V = T_{P(x,y)}$.

1. The set of all second coordinates of ordered pairs in V can be expressed as $S = \{y : \exists x, (x,y) \in V\}$. Given any b,

$$\begin{aligned} b \in T_{\exists x P(x,y)} &\Leftrightarrow \exists x P(x,b) \\ &\Leftrightarrow \exists x, (x,b) \in V \\ &\Leftrightarrow b \in S. \end{aligned}$$

Therefore, $T_{\exists x P(x,y)} = S$.

2. Given any k,

$$\begin{aligned} k \in T_{\forall x P(x,y)} &\Leftrightarrow \forall x P(x,k) \\ &\Leftrightarrow \forall x \in D_x, (x,k) \in T_{P(x,y)} \\ &\Leftrightarrow \forall x \in D_x, (x,k) \in V \\ &\Leftrightarrow D_x \times \{k\} \subseteq V \\ &\Leftrightarrow k \in \{b : b \in D_y, D_x \times \{b\} \subseteq V\}. \end{aligned}$$

Therefore, $T_{\forall x \in D_x P(x,y)} = \{b : b \in D_y, D_x \times \{b\} \subseteq V\}$.

□

Solution 30: Recall that $T_P \subseteq T_Q$ is equivalent to $P \Rightarrow Q$. Suppose $x \in D_x$, $y \in D_y$, and $(x + y \leq 1) \wedge (y - x \leq 1)$. We prove $P \Rightarrow Q$ in the following.

1. From $x + y \leq 1$:

	steps	*reasons*
(1)	$x + y \leq 1$	given
(2)	$-1 \leq x$	$x \in D_x$
(3)	$0 \leq y$	$y \in D_y$
(4)	$-1 \leq x + y$	(2) + (3)
(5)	$-1 \leq x + y \leq 1$	(1) + (4)
(6)	$(x + y)^2 \leq 1$	math (5)
(7)	$x^2 + 2xy + y^2 \leq 1$	math (6)

2. $y - x \leq 1$

	steps	*reasons*
(1)	$y - x \leq 1$	given
(2)	$0 \leq y$	$y \in D_y$
(3)	$1 \geq x$	$x \in D_x$
(4)	$-1 \leq -x$	(3) $\times -1$
(5)	$-1 \leq y - x \leq 1$	(2) + (4)
(6)	$(y - x)^2 \leq 1$	math (5)
(7)	$x^2 - 2xy + y^2 \leq 1$	math (6)

From $x + y \leq 1$ and $y - x \leq 1$ we have derived

$$x^2 + 2xy + y^2 \leq 1, \tag{2.5}$$
$$x^2 - 2xy + y^2 \leq 1. \tag{2.6}$$

Take (5) + (6), we have

$$2x^2 + 2y^2 \leq 2 \quad \Longrightarrow \quad x^2 + y^2 \leq 1.$$

Therefore, we conclude that $x^2 + y^2 \leq 1$, i.e.

$$((x + y \leq 1) \wedge (y - x \leq 1)) \Rightarrow (x^2 + y^2 \leq 1).$$

□

Solution 31: Recall the definition of subset. We say that S is a subset of X if and only if for all x, if x is in S, then x is in X. We may rewrite the definition as

$$S \subseteq X \quad \text{iff} \quad (x \in S) \Rightarrow (x \in X).$$

Please note that we are using "$\Rightarrow$" in the above definition instead of "$\rightarrow$", i.e., S is a subset of X if and only if $(x \in S \rightarrow\ x \in X)$ is a tautology. Therefore, it is legitimate not to express "for all" explicitly in the second definition.[1]

In order to easily find the error in this problem, let us rewrite the definition of subset without omitting the universal quantifier.

$$S \subseteq X \quad \text{iff} \quad \forall x(x \in S \rightarrow x \in X).$$

Therefore, $S \subseteq (A \cup B)$ iff

$$\begin{array}{lll}
\forall x(x \in S \rightarrow x \in (A \cup B)) & & (1) \\
\quad \Rightarrow & \forall x(x \in S \rightarrow (x \in A \text{ or } x \in B)) & (2) \\
\quad \Rightarrow & \forall x[(x \in S \rightarrow x \in A) \text{ or } (x \in S \rightarrow x \in B)] & \\
\quad \not\Rightarrow & \forall x(x \in S \rightarrow x \in A) \text{ or } \forall x(x \in S \rightarrow x \in B) & (3) \\
\quad \Rightarrow & (S \subseteq A \text{ or } S \subseteq B). & (4)
\end{array}$$

Step (3) in the original proof is incorrect.

We may consider an easy example to see why (2) $\not\Rightarrow$ (3). Let S, A, and B be sets defined as the following.

[1] Please bear in mind that, in mathematics, if the universe is clear or of no importance in the discourse, it is not uncommon to discard the universal quantifier *if it is the outermost quantifier* in a predicate.

$$S = \text{all children},\ A = \text{all boys},\ B = \text{all girls}.$$

It is clear that (2) is true, which means "for all x, if x is a child, then x is either a boy or a girl." However, (3) is not true, which means "either all children are boys, or all children are girls." □

Solution 32:

$$\begin{aligned}
\neg\forall x\exists y[(x \vee y) \to z] &\Leftrightarrow \exists x\neg\exists y[(x \vee y) \to z] \\
&\Leftrightarrow \exists x\forall y\neg[(x \vee y) \to z] \\
&\Leftrightarrow \exists x\forall y[(x \vee y) \wedge \neg z].
\end{aligned}$$

□

Solution 33: To find the negations of the given formulas, we use the basic rules and De Morgan's laws. Recall that $\neg\exists x p(x) \equiv \forall x\neg p(x)$ and $\neg\forall x p(x) \equiv \exists x\neg p(x)$ for any predicate $p(x)$. Thus,

1. $\neg\forall x\ \forall y\ [(x > y) \to (x - y > 0)]$

$$\begin{aligned}
&\Leftrightarrow \exists x\ \neg\forall y\ [(x > y) \to ((x - y) > 0)] \\
&\Leftrightarrow \exists x\ \exists y\ \neg[(x > y) \to ((x - y) > 0)] \\
&\Leftrightarrow \exists x\ \exists y\ [(x > y) \wedge \neg((x - y) > 0)] \\
&\Leftrightarrow \exists x\ \exists y\ [(x > y) \wedge ((x - y) \leq 0)] \\
&\Leftrightarrow \exists x\ \exists y\ [(x > y) \wedge (x \leq y)].
\end{aligned}$$

2. $\neg\forall x\ \forall y\ [(x < y) \to \exists z(x < z < y)]$

$$\begin{aligned}
&\Leftrightarrow \exists x\ \neg\forall y\ [(x < y) \to \exists z\ (x < z < y)] \\
&\Leftrightarrow \exists x\ \exists y\ \neg[(x < y) \to \exists z\ (x < z < y)] \\
&\Leftrightarrow \exists x\ \exists y\ [(x < y) \wedge \neg\exists z\ (x < z < y)] \\
&\Leftrightarrow \exists x\ \exists y\ [(x < y) \wedge \forall z\ \neg(x < z < y)] \\
&\Leftrightarrow \exists x\ \exists y\ [(x < y) \wedge \forall z\ ((z \leq x) \vee (y \leq z))] \\
&\Leftrightarrow \exists x\ \exists y\ \forall z\ [(x < y) \wedge ((z \leq x) \vee (y \leq z))] \\
&\Leftrightarrow \exists x\ \exists y\ \forall z\ [((x < y) \wedge (z \leq x)) \vee ((x < y) \wedge (y \leq z))] \\
&\Leftrightarrow \exists x\ \exists y\ \forall z\ [(z \leq x < y) \vee (x < y \leq z)].
\end{aligned}$$

□

Solution 34: Let the domain D_x be the set of all people. Define the following predicates.

$PB(x)$: x is a policeman in this beat.
$SC(x)$: x eats with our cook.
$ML(x)$: x is a man with long hair.
$PE(x)$: x is a poet.
$PR(x)$: x has been in prison.
$CC(x)$: x is our cook's cousin.
$HC(x)$: x is her cousin.
$LC(x)$: x loves cold mutton.
a : Amos Judd, who is a man.

Note: a is a constant in D. We don't think it is necessary to define another predicate to test if x is a man. Thus, we simply assume Amos Judd is a man.

Now, we can rewrite the premises in term of the predicates above.

$$\begin{array}{ll} p1: & \forall x[PB(x) \to SC(x)]. \\ p2: & \neg\exists x[ML(x) \wedge \neg PE(x)] \\ p3: & \neg PR(a). \\ p4: & \forall x[CC(x) \to LC(x)]. \\ p5: & \forall x[PE(x) \to PB(x)]. \\ p6: & \forall x[SC(x) \to HC(x)]. \\ p7: & \forall x[\neg ML(x) \to PR(x)]. \end{array}$$

And, we know that $p2$ is equivalent to the follows.

$$\begin{aligned} \neg\exists x[ML(x) \wedge \neg PE(x)] &\equiv \forall x \neg[ML(x) \wedge \neg PE(x)] \\ &\equiv \forall x[\neg ML(x) \vee PE(x)] \\ &\equiv \forall x[ML(x) \to PE(x)]. \end{aligned}$$

Since $a \in D$ and all predicates above except $P3$ are quantified with universal quantifiers, we can apply the universal specification rule and replace the variable x by a to obtain propositions with truth values T as follows.

$$\begin{array}{ll} p_1: & PB(a) \to SC(a). \\ p_2: & ML(a) \to PE(a). \\ p_3: & \neg PR(a). \\ p_4: & CC(a) \to LC(a). \\ p_5: & PE(a) \to PB(a). \\ p_6: & SC(a) \to HC(a). \\ p_7: & \neg ML(a) \to PR(a). \end{array}$$

We have

steps		reasons
1	$\neg PR(a) \rightarrow ML(a)$	p_7, Contrapositive
2	$ML(a)$	$1, p_3$, Modus Ponens
3	$PE(a)$	$2, p_2$, Modus Ponens
4	$PB(a)$	$3, p_5$, Modus Ponens
5	$SC(a)$	$4, p_1$, Modus Ponens
6	$HC(a)$	$5, p_6$, Modus Ponens

Now, we can transfer the propositions back to English, which will tell us some facts about Amos Judd.

$ML(a)$: Amos Judd is a man with long hair.
$PE(a)$: Amos Judd is a poet.
$PB(a)$: Amos Judd is a policeman on this beat.
$SC(a)$: Amos Judd eats with our cook.
$HC(a)$: Amos Judd is her cousin.

We do not know if Amos Judd is our cook's cousin, and we do not know if Amos Judd loves cold mutton. □

Solution 35: Let D_x be the set of all living things.

Define $P(x)$: x is a pig.
$W(x)$: x has wings.

The following predicates are equivalent.

$$\begin{aligned} & \neg\exists x[P(x) \wedge W(x)] \\ \iff & \forall x \neg[P(x) \wedge W(x)] \\ \iff & \forall x[\neg P(x) \vee \neg W(x)] \\ \iff & \forall x[P(x) \rightarrow \neg W(x)]. \end{aligned}$$

We can choose any one of them as the answer. □

Solution 36: The following conclusion is logically incorrect.

Premises: All soldiers can march.
Some babies are not soldiers.

Conclusion: Some babies cannot march.

To see this, let us define the following predicates,

$S(x)$: x is a soldier.
$B(x)$: x is a baby.
$M(x)$: x can march.

We can restate our question as: Is the following inference valid?

$$\begin{array}{c} \forall x(S(x) \rightarrow M(x)) \\ \exists x(B(x) \wedge \neg S(x)) \\ \hline \exists x(B(x) \wedge \neg M(x)) \end{array}$$

From the first premise and by the rule of universal specification we have, for any a,

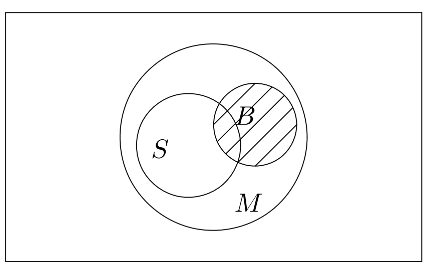
$$S(a) \rightarrow M(a),$$

but we cannot say

$$\neg S(a) \rightarrow \neg M(a)$$

because, unlike the contrapositive, the inverse is not an equivalence. Therefore, although from the premises we know that there are some babies who are not soldiers, we cannot use Modus Ponens to conclude that those babies cannot march. In other words, if one is not a soldier, that doesn't mean one cannot march. It is possible that all babies can march, while some of them are not soldiers.

To see this more clearly, consider the following Venn diagram.



Where M is the set of all creatures that can march, B is the set of babies, and S is the set of all soldiers. We interpret $x \in S$ as $S(x)$ is true, likewise for $x \in M$.

Recall $S \subseteq M \iff \forall x(x \in S \to x \in M)$. In the Venn diagram $S \subseteq M$, which gives the first premise: $\forall x(S(x) \to M(x))$.

The diagram shows $S - B \neq \emptyset$. Therefore, there are some elements that are in B but not in S, which is the second premise: $\exists x(B(x) \wedge \neg S(x))$.

However, the Venn diagram also shows that the conclusion is incorrect, because B is a subset of M, i.e., there is no baby that can't march. □

Solution 37: Let $D_x = \mathbf{N}$ and $D_y = \mathbf{N}^0$, and define two variable predicate P as

$$P(x, y) = x \text{ divides } y.$$

1. $\forall y P(1, y) = T$.
2. $\forall x P(x, 0) = T$.
3. $\forall x P(x, x) = T$.
4. $\forall y \exists x P(x, y) = T$.
   Given any number y, there exists x, say $x = 1$, such that x divides y.
5. $\exists y \forall x P(x, y) = T$. Such a y is 0.
6. $\forall x \forall y[(P(x, y) \wedge P(y, x)) \to (x = y)] = T$.
   Given any x and y, suppose that $(P(x, y) \wedge P(y, x))$ is true.
   $$\begin{aligned} P(x, y) &\Rightarrow y = ax, a \in \mathbf{N}^0; \\ P(y, x) &\Rightarrow x = by, b \in \mathbf{N}. \end{aligned}$$
   Thus, $x = by = b(ax) = abx$, and hence $ab = 1$. Because both a and b are nonnegative integers, we know that $a = b = 1$. Therefore, $x = y$.
7. $\forall x \forall y \forall z[(P(x, y) \wedge P(y, z)) \to P(x, z)] = T$.
   Given any x, y, and z, suppose that $(P(x, y) \wedge P(y, z))$ is true.
   $$\begin{aligned} P(x, y) &\Rightarrow y = ax, a \in \mathbf{N}^0; \\ P(y, z) &\Rightarrow z = by, b \in \mathbf{N}^0. \end{aligned}$$
   Thus, $z = by = b(ax) = abx$. Since $ab \in \mathbf{N}^0$, therefore, $P(x, z)$ is true.

□

Solution 38: Consider the quantified statement, $\forall x\ \exists y\ [x + y = 17]$. Let D_x and D_y denote the universes of x and y, respectively.

1. $D_x = D_y =$ the set of integers.

$$\forall x\ \exists y\ [x + y = 17]\ =\ True.$$

In this case, given any integer x, we always can find one integer $y = 17 - x$, such that $x + y = 17$. □

2. $D_x = D_y =$ the set of positive integers.

$$\forall x\ \exists y\ [x + y = 17]\ =\ False.$$

In this case, given any integer $x > 17$, we are not able to find another positive integer y, such that $x + y = 17$. □

3. $D_x =$ the set of integers and $D_y =$ the set of positive integers.

$$\forall x\ \exists y\ [x + y = 17]\ =\ False.$$

In this case, given any integer $x > 17$, we are not able to find another positive integer y, such that $x + y = 17$. □

4. $D_x =$ the set of positive integers and $D_y =$ the set of integers.

$$\forall x\ \exists y\ [x + y = 17]\ =\ True.$$

In this case, given any integer x, we always can find $y = 17 - x$, such that $x + y = 17$.

□

Solution 39: Let $f(a, b) = (a \to b) \wedge (\neg a \to \neg b)$. Let's find the truth table of f first.

a	b	$a \to b$	$\neg a \to \neg b$	$f(a, b)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The building blocks for the DNF of a propositional formula with two variables are $(a \wedge b)$, $(a \wedge \neg b)$, $(\neg a \wedge b)$, and $(\neg a \wedge \neg b)$.

a	b	$a \wedge b$	$a \wedge \overline{b}$	$\overline{a} \wedge b$	$\overline{a} \wedge \overline{b}$	$f(a, b)$	
T	T	T ✓	F	F	F	T	✓
T	F	F	T	F	F	F	
F	T	F	F	T	F	F	
F	F	F	F	F	T ✓	T	✓

Therefore, $f(a, b) = (a \wedge b) \vee (\neg a \ \wedge \neg b)$. □

Solution 40: Let $f(a, b) = (a \to b) \wedge (a \to \ \neg b)$. The truth table of f:

a	b	$a \to b$	$a \to \ \neg b$	$f(a,b)$
T	T	T	F	F
T	F	F	T	F
F	T	T	T	T
F	F	T	T	T

The associated truth table for the building blocks is:

a	b	$a \wedge b$	$a \wedge \bar{b}$	$\bar{a} \wedge b$	$\bar{a} \wedge \bar{b}$	$f(a,b)$	
T	T	T	F	F	F	F	
T	F	F	T	F	F	F	
F	T	F	F	T ✓	F	T	✓
F	F	F	F	F	T ✓	T	✓

Therefore, $f(a, b) = (\neg a \wedge b) \vee (\neg a \ \wedge \neg b)$. □

Solution 41: Remark: Since using building blocks and the truth tables to find the DNF or CNF is pretty mechanical, it should not be difficult to find the DNF and CNF in this way. Let us solve this problem by using propositional calculus. Sometimes the propositional calculus method is easier than the truth table approach, sometimes it isn't.

1. $a \longrightarrow \neg b \ = \ \neg a \vee \neg b \ = \ (\neg a \vee \neg b)$.
2. For $(a \wedge b) \vee c$:

$$\begin{aligned}(a \wedge b) \vee c &= (a \vee c) \wedge (b \vee c)\\ &= ((a \vee c) \vee F) \wedge (F \vee (b \vee c))\\ &= ((a \vee c) \vee (b \wedge \neg b)) \wedge ((a \wedge \neg a) \vee (b \vee c))\\ &= ((a \vee c \vee b) \wedge (a \vee c \vee \neg b)) \wedge ((a \vee b \vee c) \wedge (\neg a \vee b \vee c))\\ &= (a \vee b \vee c) \wedge (a \vee \neg b \vee c) \wedge (\neg a \vee b \vee c).\end{aligned}$$

Note: Why can we insert F without changing the value of the formula of the problem? How about the " $\wedge$ " case? □

Solution 42: Let $f = a \wedge (b \leftrightarrow c)$. We will use the shortcut method to find the CNF of f and propositional calculus to find the DNF.

1. Shortcut Method: First, we find the truth table for f.

a	b	c	$b \leftrightarrow c$	$a \wedge (b \leftrightarrow c)$
T	T	T	T	T
T	T	F	F	F ✓
T	F	T	F	F ✓
T	F	F	T	T
F	T	T	T	F ✓
F	T	F	F	F ✓
F	F	T	F	F ✓
F	F	F	T	F ✓

Consequently, the falsity set of f is:

$$F_f = \{(T,T,F), (T,F,T), (F,T,T), (F,T,F), (F,F,T), (F,F,F)\}.$$

Thus, the CNF of f is

$$(\overline{a}, \overline{b}, c) \wedge (\overline{a}, b, \overline{c}) \wedge (a, \overline{b}, \overline{c}) \wedge (a, \overline{b}, c) \wedge (a, b, \overline{c}) \wedge (a, b, c).$$

2. Propositional Calculus:

$$\begin{aligned} a \wedge (b \leftrightarrow c) &= a \wedge ((b \wedge c) \vee (\neg b \wedge \neg c)) \\ &= (a \wedge (b \wedge c)) \vee (a \wedge (\neg b \wedge \neg c)) \\ &= (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c). \end{aligned}$$

The last formula above is the DNF of f.

□

Solution 43: In general, "if ... then ..." can be translated into a logical implication ($\rightarrow$), and "there is" or "there are" should use an existential quantifier ($\exists$). We need a predicate for a property described. "And," "or," and "all" used in English sentences are equal to $\wedge, \vee$, and $\forall$, respectively, in the corresponding mathematical expressions. With this in mind, let's examine the statement from the number theory:

> "*For every integer n bigger than 1, there is a prime strictly between n and $2n$.*"

Let's move one step forward:

> "*For all integers, if the integer n is bigger than 1, then there is a number which is a prime and strictly between n and $2n$.*"

Let D_x be the set of all integers, and define predicate $P(x)$ as

$$P(x) : x \text{ is prime.}$$

1. We can obtain a logical sentence from the statement above:

$$\forall n[(n > 1) \rightarrow \exists x(P(x) \wedge (n < x < 2n))]. \tag{2.7}$$

2. The negation of (7) is:

$$\begin{aligned}
&\neg\forall n((n > 1) \rightarrow \exists x(P(x) \wedge (n < x < 2n))) \\
&\quad\Leftrightarrow \quad \exists n\neg((n > 1) \rightarrow \exists x(P(x) \wedge (n < x < 2n))) \\
&\quad\Leftrightarrow \quad \exists n\neg(\neg(n > 1) \vee \exists x(P(x) \wedge (n < x < 2n))) \\
&\quad\Leftrightarrow \quad \exists n((n > 1) \wedge \neg\exists x(P(x) \wedge (n < x < 2n))) \\
&\quad\Leftrightarrow \quad \exists n((n > 1) \wedge \forall x\neg(P(x) \wedge (n < x < 2n))) \\
&\quad\Leftrightarrow \quad \exists n((n > 1) \wedge \forall x(\neg P(x) \vee \neg(n < x < 2n))) \\
&\quad\Leftrightarrow \quad \exists n((n > 1) \wedge \forall x(P(x) \rightarrow \neg(n < x < 2n))) \\
&\quad\Leftrightarrow \quad \exists n((n > 1) \wedge \forall x(P(x) \rightarrow \neg((n < x) \wedge (x < 2n)))) \\
&\quad\Leftrightarrow \quad \exists n((n > 1) \wedge \forall x(P(x) \rightarrow (\neg(n < x) \vee \neg(x < 2n)))) \\
&\quad\Leftrightarrow \quad \exists n((n > 1) \wedge \forall x(P(x) \rightarrow ((x \leq n) \vee (x \geq 2n))))
\end{aligned}$$

□

Solution 44: We will use two different methods to prove this result. The first method uses the contradiction argument, and the second one will check *all* the possible cases to show that the result is valid for all of them.

Let D_x be the set of integers. Define

$$P(x) : x \text{ is even.}$$

We can restate the given result as the predicate,

$$P(xy) \longrightarrow (P(x) \vee P(y)). \tag{2.8}$$

Method 1: By way of contradiction, assume that

$$P(xy) \wedge \neg(P(x) \vee P(y)). \tag{2.9}$$

By De Morgan's law, (9) is equivalent to

$$P(xy) \wedge \neg P(x) \wedge \neg P(y). \tag{2.10}$$

If x and y are not even, then they are odd and can be expressed as

$$x = 2k + 1, y = 2q + 1$$

for some integers k and q. Thus,

$$\begin{aligned} xy &= (2k+1)(2q+1) \\ &= 4kq + 2k + 2q + 1 \\ &= 2(2kq + k + q) + 1 \end{aligned}$$

Because k and q are both integers, we know that $2kq + k + q$ is an integer too. Thus, xy is not even, i.e., $P(xy)$ is false. This contradicts our assumption that $P(xy)$ is true. Therefore, (8) is correct. □

Method 2: Given any two integers, we have three cases: 1. both are even, 2. both are odd, 3. one is odd and the other one is even. We want to show that, in each case, (8) is correct.

1. Both are even: Let $x = 2k, y = 2q$ for some integers k and q.

$$\begin{aligned} & P(xy) \rightarrow (P(x) \vee P(y)) \\ \equiv \;& P(4kq) \rightarrow (P(2k) \vee P(2q)) \\ \equiv \;& T \rightarrow (T \vee T) \\ \equiv \;& T \rightarrow T \\ \equiv \;& T \end{aligned}$$

2. Both are odd: Let $x = 2k + 1, y = 2q + 1$ for some integers k and q.

$$\begin{aligned} & P(xy) \rightarrow (P(x) \vee P(y)) \\ \equiv \;& P(4kq + 2k + 2q + 1) \rightarrow (P(2k+1) \vee P(2q+1)) \\ \equiv \;& F \rightarrow (F \vee F) \\ \equiv \;& F \rightarrow F \\ \equiv \;& T \end{aligned}$$

3. One odd and one even: Let $x = 2k + 1, y = 2q$ for some integers k and q.

$$\begin{aligned} P(xy) &\rightarrow (P(x) \vee P(y)) \\ &\equiv P(4kq + 2q) \rightarrow (P(2k + 1) \vee P(2q)) \\ &\equiv T \rightarrow (F \vee T) \\ &\equiv T \rightarrow T \\ &\equiv T \end{aligned}$$

We have seen that, in all cases, (8) is true. Therefore, it is a correct statement for all integers.

□

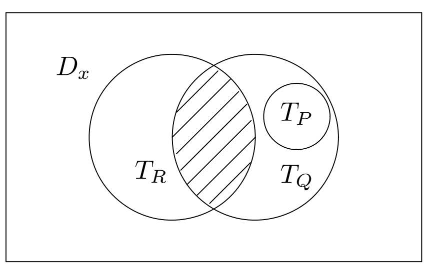
Solution 45: Let the domain D_x be the set of all people. Define the following predicates with variables over D_x.

$P(x)$: x is anxious to learn.
$Q(x)$: x works hard.
$R(x)$: x is one of those boys.

Now we can rewrite the premises and the conclusion in terms of the predicates defined above.

$$\begin{array}{rl} P1: & \forall x[P(x) \rightarrow Q(x)] \\ P2: & \exists x[R(x) \wedge Q(x)] \\ \hline C: & \exists x[R(x) \wedge P(x)] \end{array}$$

The conclusion C is not logically correct from the premises. Consider the following Venn diagram for the truth sets of predicates P, Q, and R.



$T_P \subset T_Q$ that satisfies $P1$. The shaded area, $T_R \cap T_Q$, that is not empty satisfies $P2$. $T_R \cap T_P = \emptyset$ indicates that there is no instance in the universe D_x such that both $R(x)$ and $P(x)$ are true. Therefore, the conclusion C does not follow from the premises. □

Solution 46: Let the domain D_x be the set of all people. Define the following predicates with variables over D_x.

$P(x)$:	x is a man.
$Q(x)$:	x is a soldier.
$R(x)$:	x is strong.
$S(x)$:	x is brave.

The premises and the conclusion in terms of the predicates defined above are:

$$\begin{array}{rl} P1: & \exists x[P(x) \wedge Q(x)] \\ P2: & \forall x[Q(x) \rightarrow R(x)] \\ P3: & \forall x[Q(x) \rightarrow S(x)] \\ \hline C: & \exists x[P(x) \wedge R(x) \wedge S(x)] \end{array}$$

We use two different approaches to solve this problem.

Method 1: Let T_P, T_Q, T_R, and T_S be the truth sets of the predicates defined above. From the premises we can have the following facts:

$$\begin{array}{lcl} P1 & \Longleftrightarrow & T_P \cap T_Q \neq \emptyset \\ P2 & \Longleftrightarrow & T_Q \subseteq T_R \\ P3 & \Longleftrightarrow & T_Q \subseteq T_S \end{array}$$

Based on the facts above, we want to prove the conclusion that is equivalent to

$$(T_P \cap T_R \cap T_S) \neq \emptyset.$$

From $P1$, we know there is a nonempty set α such that $T_P \cap T_Q = \alpha$.

	steps	*reasons*
1.	$\alpha \subseteq T_Q$	Definition of $\cap$
2.	$T_Q \subseteq T_R$	$P2$
3.	$\alpha \subseteq T_R$	1, 2, Definition of $\subseteq$
4.	$T_Q \subseteq T_S$	$P3$
5.	$\alpha \subseteq T_S$	1, 4, Definition of $\subseteq$
6.	$\alpha \subseteq T_P$	Definition of $\cap$
7.	$\alpha \subseteq (T_P \cap T_R \cap T_S)$	3, 5, 6, Definition of $\cap$
8.	$\alpha \neq \emptyset$	$P1$
9.	$(T_P \cap T_R \cap T_S) \neq \emptyset$	steps 7, 8

Since $(T_P \cap T_R \cap T_S) \neq \emptyset$, there must be at least one element a in the universe (domain D_x) such that $(P(a) \wedge R(a) \wedge S(a))$ is true. Thus, we can conclude that

$$\exists x(P(x) \wedge R(x) \wedge S(x)) = T,$$

i.e., conclusion C is correct. □

Method 2: Proof by using the laws of logic and inference rules for quantified predicate calculus.

	steps	*reasons*
1.	$\exists x[P(x) \wedge Q(x)]$	$P1$
2.	$P(a) \wedge Q(a)$	1, Existential Specification
3.	$\forall x[Q(x) \rightarrow R(x)]$	$P2$
4.	$Q(a) \rightarrow R(a)$	3, Universal Specification
5.	$Q(a)$	2, Conjunctive Simplification
6.	$R(a)$	4, 5, Modus Ponens
7.	$\forall x[Q(x) \rightarrow S(x)]$	$P3$
8.	$Q(a) \rightarrow S(a)$	7, Universal Specification
9.	$S(a)$	5, 8Modus Ponens
10.	$P(a)$	2, Conjunctive Simplification
11.	$P(a) \wedge R(a) \wedge S(a)$	6, 9, 10, conjunction
12.	$\exists x(P(x) \wedge R(x) \wedge S(x))$	11, Existential Generalization

□

Solution 47:

1. All members love each other.
2. There are some members who love some of the other members.
3. All members love some members.
4. There are some members who love all of the other members.

□

Solution 48: There are many alternatives for the conclusion that can be derived from the given premises. Let us just write down three of the most obvious, but not trivial, conclusions and see how to derive them logically from the premises.

We first define some needed predicates and translate the English sentences into logical formulas:

$P(x)$: x is an integer.
$Q(x)$: x is a rational number.
$R(x)$: x is a real number.

Premises:

$$P1: \forall x[P(x) \rightarrow Q(x)]$$
$$P2: R(\pi) \land \neg Q(\pi)$$

	steps	*reasons*
1.	$\forall x(P(x) \rightarrow Q(x))$	$P1$
2.	$R(\pi) \land \neg Q(\pi)$	$P2$
3.	$P(\pi) \rightarrow Q(\pi)$	1, Universal Specification
4.	$\neg Q(\pi) \rightarrow \neg P(\pi)$	3, Contrapositive
5.	$\neg Q(\pi)$	2, Conjunctive Simplification
• 6.	$\neg P(\pi)$	4, 5, Modus Ponens
7.	$R(\pi)$	2, Conjunctive Simplification
8.	$R(\pi) \land \neg P(\pi)$	6, 7, Conjunction
• 9.	$\exists x(R(x) \land \neg P(x))$	8, Existential Generalization
10.	$\neg P(\pi) \lor \neg R(\pi)$	6, Disjunctive Amplification
11.	$\neg[P(\pi) \land R(\pi)]$	10, De Morgan's Law
12.	$\exists x \neg(P(x) \land R(x))$	11, Existential Generalization
• 13.	$\neg\forall x(P(x) \land R(x))$	12, Logical Equivalence

- In step 6, π *is not an integer.*

- In step 9, *there is a real but not rational number.*

- In step 13, *not all numbers are both integer and real.*

□

Solution 49: Let m denote Margaret. Define

$$\begin{aligned} P(x): &\quad x \text{ is a librarian.} \\ Q(x): &\quad x \text{ knows the system.} \end{aligned}$$

We want to use the rule of universal specification and Modus Ponens. If the unknown premise is "Margaret is a librarian," then we can have the following inference.

$$\begin{array}{rl} P1: & \forall x[P(x) \to Q(x)] \\ P2: & P(m) \\ \hline C: & Q(m) \end{array}$$

steps		*reasons*
1.	$\forall x(P(x) \to Q(x))$	$P1$
2.	$P(m)$	$P2$
3.	$P(m) \to Q(m)$	1, Universal Specification
4.	$Q(m)$	2, 3, Modus Ponens

Where $P(m)$ means: <u>*Margaret is a librarian*</u>.

□

Solution 50: Prove $\exists x[P(x) \vee Q(x)] \iff \exists x P(x) \vee \exists x Q(x)$.

For $\Rightarrow$ direction, assume $\exists x[P(x) \vee Q(x)]$.

steps		reasons
1.	$\exists x[P(x) \vee Q(x)]$	Assumption
2.	$P(a) \vee Q(a)$	Existential Specification, 1
3.	$\exists x P(x) \vee Q(a)$	Existential Generalization, 2
4.	$\exists x P(x) \vee \exists x Q(x)$	Existential Generalization, 3

For $\Leftarrow$ direction, assume $\exists x P(x) \vee \exists x Q(x)$,

	steps	*reasons*
1.	$\exists x P(x) \vee \exists x Q(x)$	Assumption
2.	$P(a) \vee \exists x Q(x)$	1, Existential Specification
3.	$P(a) \vee Q(b)$	2, Existential Specification
4.	$[P(a) \vee Q(b)] \vee [P(b) \vee Q(a)]$	3, Disjunctive Amplification
5.	$[P(a) \vee Q(a)] \vee [P(b) \vee Q(b)]$	4, Associative, Commutative
6.	$\exists x[P(x) \vee Q(x)] \vee \exists x[P(x) \vee Q(x)]$	5, Existential Generalization
7.	$\exists x[P(x) \vee Q(x)]$	$A \vee A \equiv A$

This completes the proof.

Note: Be very careful in step 3 of the proof of $\Leftarrow$ direction. We have to use two different symbols, a and b, because they are specified from two different existential quantifiers, and they may or may not be equal.

□

Solution 51: Prove $\forall x[P(x) \wedge Q(x)] \iff \forall x P(x) \wedge \forall x Q(x)$.

For $\Rightarrow$ direction, by way of contradiction, assume

$$\forall x[P(x) \wedge Q(x)] \quad \wedge \quad \neg[\forall x P(x) \wedge \forall x Q(x)].$$

	steps	*reasons*
1.	$\neg[\forall x P(x) \wedge \forall x Q(x)]$	Assumption
2.	$\neg[P(a) \wedge \forall x Q(x)]$	1, Universal Specification
3.	$\neg[P(a) \wedge Q(a)]$	2, Universal Specification
4.	$\exists x \neg[P(x) \wedge Q(x)]$	3, Existential Specification
5.	$\neg\forall x[P(x) \wedge Q(x)]$	4, Logical Equivalence

The conclusion in step 5 contradicts our assumption.

Note: We choose the same a in steps 2 and 3.

For $\Leftarrow$ direction, by way of contradiction, assume

$$[\forall x P(x) \wedge \forall x Q(x)] \wedge \neg\forall x[P(x) \wedge Q(x)]$$

	steps	*reasons*
1.	$\neg\forall x[P(x) \wedge Q(x)]$	Assumption
2.	$\exists x \neg[P(x) \wedge Q(x)]$	1, Logical Equivalence
3.	$\neg[P(a) \wedge Q(a)]$	2, Existential Specification
4.	$\neg P(a) \vee \neg Q(a)$	3, De Morgan's law
5.	$\exists x \neg P(x) \vee \exists x \neg Q(x)$	4, Existential Generalization
6.	$\neg\forall x P(x) \vee \neg\forall x Q(x)$	5, Logical Equivalence
7.	$\neg[\forall x P(x) \wedge \forall x Q(x)]$	6, De Morgan's law

Step 7 gives a conclusion that contradicts our assumption:

$$\forall x P(x) \wedge \forall x Q(x).$$

Therefore, both directions hold. This completes the proof. □

Solution 52: To prove $\forall x P(x) \vee \forall x Q(x) \Rightarrow \forall x[P(x) \vee Q(x)]$, by contradiction, assume that

$$[\forall x P(x) \vee \forall x Q(x)] \ \wedge \ \neg\forall x[P(x) \vee Q(x)].$$

	steps	reasons
1.	$\neg\forall x[P(x) \vee Q(x)]$	Assumption
2.	$\exists x\neg[P(x) \vee Q(x)]$	1, Logical Equivalence
3.	$\neg[P(a) \vee Q(a)]$	2, Existential Specification
4.	$\neg P(a) \wedge \neg Q(a)$	3, De Morgan's law
5.	$\neg P(a)$	4, Conjunctive Simplification
6.	$\exists x\neg P(x)$	5, Existential Generalization
7.	$\neg\forall x P(x)$	6, Logical Equivalence
8.	$\neg Q(a)$	4, Conjunctive Simplification
9.	$\exists x\neg Q(x)$	8, Existential Generalization
10.	$\neg\forall x Q(x)$	9, Logical Equivalence
11.	$\neg\forall x P(x) \wedge \neg\forall x Q(x)$	7, 10, Conjunction
12.	$\neg[\forall x P(x) \vee \forall x Q(x)]$	11, De Morgan's law

Step 12 gives a conclusion that contradicts our assumption:

$$\forall x P(x) \vee \forall x Q(x).$$

This proves the theorem. □

Solution 53: To disprove $\forall x[P(x) \vee Q(x)] \Rightarrow \forall x P(x) \vee \forall x Q(x)$, we will construct a counter example.

Let the universe be $\{a, b\}$, and let p, q be two predicates with the truth values defined in the following table.

	a	b
$P(x)$	T	F
$Q(x)$	F	T

$$\begin{aligned} \forall x[P(x) \vee Q(x)] &= [P(a) \vee Q(a)] \wedge [P(b) \vee Q(b)] \\ &= [T \vee F] \wedge [F \vee T] \\ &= T \wedge T \\ &= T \end{aligned}$$

$$\begin{aligned} \forall x P(x) \vee \forall x Q(x) &= [P(a) \wedge P(b)] \vee [Q(a) \wedge Q(b)] \\ &= F \vee F \\ &= F \end{aligned}$$

Therefore, $\forall x[P(x) \vee Q(x)] \not\Rightarrow \forall x P(x) \vee \forall x Q(x)$. □

Solution 54: We already know that $\sqrt{2}$ is irrational. Now, consider the number $\sqrt{2}^{\sqrt{2}}$. We don't know whether $\sqrt{2}^{\sqrt{2}}$ is irrational or not. But it is certain that there are only two cases.

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. Let $a = \sqrt{2}$ and $b = \sqrt{2}$. Then both a and b are irrational and a^b is rational.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. Let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$.

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2.$$

Therefore, both a and b are irrational and a^b is rational.

In both cases we can claim that there are a and b, where a and b are irrational and a^b is rational. □