Problems on Discrete Mathematics[1]

Chung-Chih Li[2]
Kishan Mehrotra[3]

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[2]Illinois State University, Normal, Illinois. cli2@ilstu.edu
[3]Syracuse University, Syracuse, New York. kishan@ecs.syr.edu

Chapter 3

Mathematical Induction

To develop the skill of correct thinking is in the first place
to learn what you have to disregard.
In order to go on, you have to leave out;
this is the essence of effective thinking.

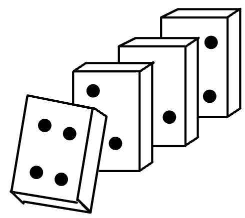
– Kurt Gödel

3.1 Concepts

Mathematical induction is one of the most important and powerful techniques for verifying mathematical statements. Many complicated mathematical theorems about integers can be proved easily by mathematical induction. It is easy because the frame of the proof is unique and the underlying idea of mathematical induction is intuitively understandable.

Think of infinitely many dominoes lined up. The proof by mathematical induction is like the domino effect illustrated in the following figure:

If the first domino falls, and for any n if the fall of the n^{th} domino would result in the knockdown of the $(n+1)^{st}$ domino, then any domino in the line will fall.



3.1.1 Necessary Conditions of Using Mathematical Induction

In the domino analogue above, to be able to (1) line up the dominoes, and (2) identify the first domino in the line are two necessary conditions for knocking down all of the dominoes. The two conditions are exactly the same necessary conditions for being able to make use of mathematical induction.

Suppose a given mathematical statement is about a property over the domain S. If all elements in S can be *well ordered*, i.e., (1) we can enumerate all elements in S one by one, and (2) we can identify the first element in the enumeration, then mathematical induction is very likely to be the approach for verifying the statement.

In particular, any well-ordered set S can be represented as

$$S = \{s_0, s_1, s_2, \ldots\}.$$

In this representation, (1) for any element a in S, there is $n \in \mathbf{N}$ such that $a = s_n$ (all dominoes have been lined up properly), and (2) s_0 is identified, which is called the *least element* (s_0 serves as the first domino in the line to fall.)

Example 3.1 Sets such as natural numbers, integers, even numbers, odd numbers, multiples of 5, etc., are typical well-ordered sets that are often encountered in mathematics. They can be represented as:

$$\begin{aligned}\text{Natural numbers} &: \{1,2,3,4,\ldots\};\\ \text{Integers} &: \{0,1,-1,2,-2,3,-3,\ldots\};\\ \text{Even numbers} &: \{0,2,-2,4,-4,\ldots\};\\ \text{Odd numbers} &: \{1,-1,3,-3,5,-5,\ldots\};\\ \text{Multiples of 5} &: \{0,5,-5,10,-10,\ldots\}.\end{aligned}$$

For more advanced mathematical inductive proof, the domain set may be rational numbers, prime numbers, or any other well-ordered sets.

3.1.2 The Underlying Theory of Mathematical Induction

The underlying theory of mathematical induction is very simple. It is nothing but repeated application of the logical rule, Modus Ponens. Suppose

$$S = \{s_0, s_1, s_2, \ldots\},$$

and we have proved that

$$[P(s_0) = T] \text{ and } [\forall n \in \mathbf{N} P(s_n) \rightarrow P(s_{n+1})].$$

Given any $i \in \mathbf{N}$, we can claim that $P(s_i) = T$ by the following inferences.

	steps	*reasons*
1.	$P(s_0)$	proved
2.	$P(s_0) \rightarrow P(s_1)$	proved
3.	$P(s_1)$	1,2, Modus Ponens
4.	$P(s_1) \rightarrow P(s_2)$	proved
5.	$P(s_2)$	3,4, Modus Ponens
6.	$P(s_2) \rightarrow P(s_3)$	proved
7.	$P(s_3)$	5,6, Modus Ponens
	$\vdots$	$\vdots$
	$P(s_{i-1})$	Modus Ponens
	$P(s_{i-1}) \rightarrow P(s_i)$	proved
	$P(s_i)$	Modus Ponens

Therefore, we can claim that for all a in S, $P(a)$ is a true statement.

Comment: In most problems, to prove $P(s_0) = T$ is trivial. The main task is to prove that $P(s_n) \rightarrow P(s_{n+1})$ for any n.

3.1.3 Mathematical Induction of the First Form (Weak Induction)

Suppose that the universal set (domain) is the set of non-negative integers, and $P(n)$ is any property of non-negative integers. We wish to prove that $P(n)$ is true for all $n = 0, 1, 2, \ldots$; i.e., $\forall n P(n)$. Then the following procedure can be applied.

Step 1: Prove that $P(0)$ is true. This step is known as the *basis step*, and the proved result, $P(0) = T$, is called the *basis of induction.*

Step 2: Let n be an arbitrary fixed integer, and assume that $P(n)$ is true. This assumption is called the *weak inductive hypothesis.*

Step 3: Use the assumption in step 2 to prove that $P(n+1)$ is true. This step is known as the *inductive step.*

If we can prove the basis in step 1 and the implication in step 3, then we can claim that $P(n)$ is true for all $n = 0, 1, 2, \ldots$. This method of proof is known as the *mathematical induction of the first form* or the *weak induction.*

Comment: The above proof procedure is an application of the following rule of inference,

$$\begin{array}{ll} 1. & P(0) \\ 2. & \forall n[P(n) \longrightarrow P(n+1)] \\ \hline 3. & \forall n P(n), \end{array}$$

where the first assertion, $P(0)$, is proved in step 1, and the second assertion, $\forall n[P(n) \longrightarrow P(n+1)]$, is shown by picking an arbitrary value of n for $P(n)$ in step 2 and by the implication proved in step 3.

Comment: One should not be confused by the statement in step 2 "let n be an arbitrary fixed integer, and assume that $P(n) = T$" and the goal "for all n $P(n) = T$" that we want to prove. The letter n in step 2 denotes an instance in the domain, and the letter n in the statement $\forall n P(n)$ is a variable that ranges over the domain. After step 2, the instance n is fixed. The following modification makes the difference explicit, but it also makes the proof awkward, and hence, for simplicity, most textbooks do not use it.

Step 2: For the inductive hypothesis, we assume that

$$P(n) = T \text{ when } n = i \text{ for some } i \text{ in the domain.}$$

Step 3: In the inductive step, we prove that

$$P(i) \longrightarrow P(i+1).$$

Then we conclude that for all n in the domain, $P(n) = T$.

Comment: In some instances, a weak inductive hypothesis cannot provide sufficient ground to prove that $P(n+1)$ is true in step 3. We need a stronger inductive hypothesis that is introduced in the next subsection.

3.1.4 Mathematical Induction of the Second Form (Strong Induction)

Suppose that the universal set (domain) is the set of non-negative integers, and $P(n)$ is any property of non-negative integers. We wish to prove that $P(n)$ is true for all $n = 0, 1, 2, \ldots$; i.e., $\forall n P(n)$. The following procedure can be applied.

Step 1: Prove that $P(0)$ is true. This step is known as the *basis step*, and the proved result $P(0) = T$ is called the *basis of induction.*

Step 2: Let n be an arbitrary fixed integer in the domain, and assume that $P(0), P(1), \ldots,$ and $P(n)$ are true. This assumption is called the *strong inductive hypothesis.*

Step 3: Use the assumption to prove that $P(n+1)$ is true. This step is known as the *inductive step.*

If we can prove the basis in step 1 and the implication in step 3, then we can claim that that $P(n)$ is true for all $n = 0, 1, 2, \ldots$. This method of proof is known as the *mathematical induction of the second form,* or the *strong induction.*

Comment: The above proof procedure is an application of the following rule of inference,

$$\begin{array}{ll} 1. & P(0) \\ 2. & \forall n[(P(0) \wedge P(1) \wedge \cdots \wedge P(n)) \longrightarrow P(n+1)] \\ \hline 3. & \forall n P(n) \end{array}$$

Comment: Sometimes we prove that $P(n)$ is true for $n \in S$, where $S = \{s_0, s_1, s_2, \ldots\}$, and s_0 may not be equal to 0, or in some cases S may not be a set of numbers at all. If that is the case, the basis step changes to:

Step 1: Show that $P(s_0)$ is true.

Comment: In some problems, one may claim that $P(n)$ is true for all $n \in \{0, 1, 2, \ldots\}$, but the basis is not $P(0)$, but $P(1)$ or $P(2)$. It is important to recognize this property; otherwise, incorrect proofs are obtained. See problem 17 in the problem section of this chapter and its note on page 33.

3.2 Mathematical Induction and Recursive Definition

Mathematical induction and recursive definitions are intimately related. They should be viewed as two sides of the same coin. Recursion is a very useful apparatus for defining sets, functions,[1] and the programming syntax.[2]

3.2.1 Recursive Definitions for Functions

Theoretically, any *computable* function can be defined recursively. In the following we take the point of view that the argument of the function f takes values in the set of non-negative integers. In general, recursively defined functions have an infinitely large domain set.

A function f can be recursively defined as follows:

1: Define the value of the function at a few points. For example, $f(0), f(1)$ are specified. Such values are called the *initial values*.

2: Define the value of the function at $n+1$ in terms of $f(0)$, $f(1)$, $\ldots$, $f(n)$, and n itself.

3: Write a closing statement that in most cases reads "Steps 1 and 2 are the only two steps that define the function f."

Comment: The essence of recursive definition is its simplicity that helps us to understand the function being defined, but not its efficiency when we are asked to actually find the value of the function. We prefer to compute a function by using its *closed-form* formula instead of by using its recursive definition directly. Mathematical induction has nothing to do with finding a closed-form formula for a given function,[3] but it is a powerful technique for verifying that a given closed-form formula is a correct one for the recursively defined function.

Example 3.2 Let f be a function taking a non-negative integer as its argument, and be recursively defined as follows.

1: $f(0) = 0$.

[1] Basic concepts associated with functions are considered in Chapter 5.

[2] The definition 2.6 in Chapter 2 is an example of recursive definitions for the syntax of well-formed formulas.

[3] Chapter 8 will discuss details about solving recurrence relations.

2: For all $n \geq 0$, $f(n+1) = f(n) + (n+1)$.

We note that

$$\begin{aligned} f(0) &= 0, \\ f(1) &= f(0) + 1 = 0 + 1 = 1, \\ f(2) &= f(1) + 2 = 0 + 1 + 2 = 3, \\ f(3) &= f(2) + 3 = 0 + 1 + 2 + 3 = 6, \\ &\vdots \end{aligned}$$

and conjecture that $f(n)$ is the summation of the first n natural numbers, i.e., the closed-form of f is given as

$$f(n) = \frac{n(n+1)}{2}, \quad \text{for all } n \geq 0. \tag{3.1}$$

We can use mathematical induction to prove that equality (1) is correct.[4]

Inductive Basis: The initial value of f serves as the basis of the induction. In particular, we prove that $f(0) = \frac{0\times(0+1)}{2}$. It's clear that the equality (1) holds when $n = 0$, and hence the basis holds.

Note: $f(0) = 0$ is given as a part of the definition of f, and we have verified it by using formula (1).

Inductive Hypothesis: Assume that given any fixed $n \geq 0$, the equality (1) holds, i.e.,

$$f(n) = \frac{n(n+1)}{2}.$$

Note: We removed the universal quantifier in the equality (1), because n is fixed after this step.

Inductive Step: In this step, we need to prove that

$$f(n+1) = \frac{(n+1)(n+2)}{2}.$$

We first use the recursive definition of f to have

$$f(n+1) = f(n) + (n+1). \tag{3.2}$$

[4] We do not explicitly define a predicate to be proved true in its domain for this example. Implicitly, the predicate is:

$$P(n) \ : \ f(n) = \frac{n(n+1)}{2},$$

and $D_n = \mathbf{N}^0$. In general, if the predicate is clear from the problem context, we can omit it to improve the compactness of the proof. Please compare the solutions to Problems 6 and 7. On the other hand, if the subject is not clear, a well-stated predicate and its domain can help us to move the first step (see Problem 32.) We will explicitly define a predicate in most solutions in Section 0.5.

Then, we use the hypothesis to replace $f(n)$ in (2) by $\frac{n(n+1)}{2}$. We have

$$\begin{aligned} f(n+1) &= f(n) + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Therefore, we can claim that for all $\forall n \geq 0, f(n) = \frac{n(n+1)}{2}$, and hence the given closed-form for f is correct. □

3.2.2 Recursive Definitions for Sets and Structural Induction

We will introduce an important variant of mathematical induction called *structural induction* in this subsection. Let us first study how to define a set by using recursive definitions. The idea is: (1) give a few initial elements for the set, and (2) specify a few rules to construct new elements by using old elements that are already in the set.

A set S can be recursively defined as follows:

1: Identify a few elements of the set S.

2: Explain how to obtain new elements of the set S from the old elements of the set.

3: Write a closing statement that in most cases reads "(1) and (2) are the only two ways to generate elements of the desired set S."

Example 3.3 Let set S be a set of strings of 1's and 0's and be recursively defined as follows:

1: $1 \in S$, $100 \in S$.

2: If $s \in S$, then $11s \in S$.

3: If $s \in S$, then $00s \in S$.

4: Nothing but strings generated according to rules 1, 2, and 3 are elements in S.

The following table shows some elements in the set and the rules used to generate each element in the table, respectively.

	elements	rules used
s_0	1	1
s_1	001	$1, 3$
s_2	1100111	$1, 2, 3, 2$
s_3	001100001	$1, 3, 3, 2, 3$
s_4	11001100001	$1, 3, 3, 2, 3, 2$
s_5	00001100001	$1, 3, 3, 2, 3, 3$

In the example above, given any element $s \in S$, according to the rules we can construct two new elements from s, i.e., $11s$ (using rule 2) and $00s$ (using rule 3). For example, if we know that $s_3 \in S$, then we can apply rules 2 and 3 to s_3 to obtain $11s_3$ and $00s_3$ which are s_4 and s_5, respectively, and they are also to be included in S. On the other hand, 1111, for example, is not a member of S because we cannot have 1111 by using the given rules.

Now, the question is, what is the relation between a recursively defined set and mathematical induction? To answer this question, let's observe the elements $s_0, \ldots, s_5$ of the set S obtained above. We find that every element has an even number of $0's$ and an odd number of $1's$. Is this observation correct in general. How can we prove it? The best technique for proving this kind of problem is mathematical induction. In general, if a set is recursively defined and we observe that the recursive definition gives a common property shared by all elements in the set, then we can use mathematical induction to prove the observation.

The difficulty is that the order of elements in the set S is no longer obvious (although S is still a well-ordered set). To overcome this problem, we introduce a variant of mathematical induction called *structural induction.* We do not visually line up the elements in the set. Instead, we imagine that the elements in the set are lined up according to the numbers of times the rules used to generate them. For example, if s is somewhere in the line, then the next elements are $00s$ and $11s$, because $00s$ and $11s$ are obtained from s by applying one of the given rules one more time.

Therefore, if a given property P is claimed to be universal in the set S, then we should be able to prove the implication:

$$\forall s \in S\ [P(s) \rightarrow P(00s) \wedge P(11s)].$$

The steps of proof by structural induction are stated below. In general, let set S be recursively defined as follows:

1. $s_0, s_1, \ldots, s_k \in S$.

2. If $s \in S$, then $r_0(s), r_1(s), \ldots, r_l(2) \in S$.

3. Only elements specified in **1** or generated by rules in **2** are elements in S.

We wish to prove that all elements in S have the property P, i.e.,

$$\forall s \in S\ [P(s) = T].$$

Then the following procedure can be applied.

Step 1: Prove that $P(s_0), P(s_1), \ldots, P(s_k)$ are all true. This step is the *basis step* of structure induction.

Step 2: Let s be any arbitrary fixed element in the set S, and assume that $P(s)$ is true. This is the *inductive hypothesis.*

Step 3: Use the assumption to prove that $P(r_0(s))$, $P(r_1(s))$, $\ldots$, $P(r_l(s))$ are all true. This step is the *inductive step.*

If we can prove the basis in step 1 and the implication in step 3, then we can claim that that $P(s)$ is true for all $s \in S$.

This method is called *structural induction*, because we are examining the structure of the elements in the set, where the structure is given by the rules. Any non-initial element can be decomposed into a few small fragments, and its property is the result of its fragments' properties.

Example 3.4 Consider the set S defined in the example on page 9. Prove by structural induction that every element in S has an even number of $0's$ and an odd number of $1's$.

Inductive Basis: It is clear that both strings "1" and "100" have an even number of $0's$ and an odd number of $1's$. Thus the inductive basis holds.

Inductive Hypothesis: Assume $s \in S$ and s has an even number of $0's$ and an odd number of $1's$.

Inductive Step: It is clear that, if the assumption is true, then both $00s$ and $11s$ have an even number of $0's$ and an odd number of $1's$.

Therefore, the observation is correct. □

3.3 Nested Induction

Nested induction is a special form of mathematical induction, by which we can prove a mathematical statement that has more than one variables involved. Nested induction is also known as *double induction* in case that the subject mathematical statement has two variables. Consider the Ackerman function

$A : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ defined as follows. For every $m, n \in \mathbf{N}$,

$$\begin{array}{rl} (i) & A(0,m) = m+1; \\ (ii) & A(n+1,0) = A(n,1); \\ (iii) & A(n+1,m+1) = A(n, A(n+1,m)). \end{array}$$

Theorem: For all $m, n \in \mathbf{N}, A(n,m) > m$.

We will prove the theorem by nested induction directly before we give its underlying theory to justify our steps. Let m and n range over $\mathbf{N}$.

Basis: $\forall m A(0,m) > m$. It plainly appears from definition (i).

Inductive Hypothesis: Fix $n \in \mathbf{N}$, assume that

$$\forall m[A(n,m) > m].$$

Inductive Steps: We want to prove that $\forall m[A(n+1,m) > m]$.

For $m = 0$, we have $A(n+1,0) = A(n,1)$ by definition (ii), and by the hypothesis, $A(n,1) > 1$, Thus, $A(n+1,0) > 1 > 0$.

For $m > 0$, we have

$$\begin{array}{rcll} A(n+1,m) & = & A(n, A(n+1,m-1)) & \text{by def } (iii) \\ & > & A(n+1,m-1) & \text{by the hypothesis} \\ A(n+1,m) & \geq & A(n+1,m-1)+1 & \\ & = & A(n, A(n+1,m-2))+1 & \text{by def } (iii) \\ & > & A(n+1,m-2)+1 & \text{by the hypothesis} \\ A(n+1,m) & \geq & A(n+1,m-2)+2 & \\ & \vdots & & \\ A(n+1,m) & \geq & A(n+1,m-m)+m & \\ & = & A(n+1,0)+m & \\ & = & A(n,1)+m & \\ & > & 1+m & \text{by the hypothesis} \end{array}$$

Therefore, $\forall m[A(n+1,m) > m]$. □

3.3.1 The underlying logic of nested induction

We use the previous example, and define two-place and one-place predicates Q and P, respectively, over natural numbers as follows.

$$\begin{array}{rcl} Q(n,m) & \triangleq & A(n,m) > m; \\ P(n) & \triangleq & \forall m Q(n,m). \end{array}$$

Thus, we can rewrite the theorem according the following logic equivalences.

$$\forall n \forall m [A(n,m) > m] \iff \forall n \forall m Q(n,m) \iff \forall n P(n).$$

To prove $\forall n P(n)$ by mathematical induction, we apply the inference rule we have been familiar with:

$$\begin{array}{l} P(0) \\ \forall n[P(n) \Rightarrow P(n+1)] \\ \hline \forall n P(n). \end{array}$$

Use the definition of P, the inference above can be rewritten as:

$$\begin{array}{l} \forall m Q(0,m) \\ \forall n[\forall m Q(n,m) \Rightarrow \forall m Q(n+1,m)] \\ \hline \forall n \forall m Q(n,m). \end{array}$$

Therefore, in the inductive proof we just shown, the inductive basis is $\forall m Q(0,m)$ and the inductive hypothesis is $\forall m Q(n,m)$.

3.4 Problems

Conventions for the rest of this chapter:

- Unless we state otherwise, n ranges over $\mathbf{N}^0$, i.e., $n \in \{0, 1, 2, 3, \ldots\}$.
- All indexing variables in $\sum$ notation are integers.
- **TH** stands for inductive hypothesis.

Problem 1: The assertion

$$\sum_{1 \le k \le n} 2^k = 2^{n+1}, \quad \forall n \ge 1$$

is incorrect. Find the mistake of the invalid proof in the following:

1. Assume that for a fixed n

$$\sum_{1 \le k \le n} 2^k = 2^{n+1}.$$

2. Add 2^{n+1} to both sides of the above equality. We have

$$\begin{aligned} \sum_{1 \le k \le n} 2^k + 2^{n+1} &= 2^{n+1} + 2^{n+1} \\ \sum_{1 \le k \le n+1} 2^k &= 2 \times 2^{n+1} \\ \sum_{1 \le k \le n+1} 2^k &= 2^{n+1+1}. \end{aligned}$$

This proves the result.

Problem 2: Consider the sequence $a_0, a_1, a_2, \ldots$, defined as:

1: $a_0 = 2$, and $a_1 = 3$.

2: For $n \ge 2$, define $a_n = 2a_{n-1} - a_{n-2}$.

We are given the following assertion:

"There is no element in the sequence equal to 1".

We present the following proof. What is wrong with the proof?

If we can prove that $a_0 \ne 1$ and the sequence is increasing, i.e., for all n, $a_{n+1} > a_n$, then we can claim that no element in the sequence can be equal to 1.

Inductive Basis: It's clear that $a_0 \neq 1$. **[The Basis Holds.]**

Inductive Hypothesis: Assume $a_{n+1} > a_n$.

Inductive Step: From the definition of the sequence and the hypothesis, we have

$$\begin{aligned} a_{n+2} &= 2a_{n+1} - a_n \\ &> 2a_{n+1} - a_{n+1} = a_{n+1}. \end{aligned}$$

Thus, $a_{n+2} > a_{n+1}$. **[The Inductive Step Holds.]**

Problem 3: Let $n \in \mathbf{N}$, and consider

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!. \tag{3.3}$$

1. Use the $\sum$ notation to express the above expression.
2. Find the formula for it.

Problem 4: Prove the formula obtained for (3) by mathematical induction.

Problem 5: Let $n \in \mathbf{N}$, and consider

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{(3n-2) \cdot (3n+1)}. \tag{3.4}$$

1. Use the $\sum$ notation to express the expression above.
2. Find the formula for it.

Problem 6: Prove the formula obtained for (4) by mathematical induction.

Problem 7: Prove that

$$\sum_{1 \leq k \leq n} k(k+1) = n(n+1)(n+2)/3.$$

Problem 8: Prove by mathematical induction that

$$\sum_{0 \leq k \leq n} 3^k = \frac{(3^{n+1} - 1)}{2}.$$

Problem 9: Prove by mathematical induction that

$$\sum_{1 \leq k \leq n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{3.5}$$

Problem 10: Prove by mathematical induction that

$$\sum_{1 \leq k \leq n} k(k-1) = \frac{1}{3}(n+1)n(n-1). \tag{3.6}$$

Problem 11: Prove by mathematical induction that

$$\sum_{1\le k\le n} k(k-1)(k-2) = \frac{1}{4}(n+1)n(n-1)(n-2). \tag{3.7}$$

Problem 12: Use the equalities (5), (6), and (7) to derive a formula for

$$\sum_{1\le k\le n} k^3. \tag{3.8}$$

Problem 13: Prove the closed form obtained for (8) by mathematical induction .

Problem 14: Prove by mathematical induction that for all positive odd n,

$$\sum_{0\le k\le n} (-2)^k = \frac{1}{3}(1-2^{n+1}).$$

Problem 15: Prove by mathematical induction that for all $n \ge 1$,

$$\sum_{1\le k\le n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

Problem 16: Prove by mathematical induction that for all $n \ge 1$,

$$\sum_{1\le k\le n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

Problem 17: Prove the following by mathematical induction :

$$\overline{\bigcup_{1\le i\le n} A_i} = \bigcap_{1\le i\le n} \overline{A_i},$$

where n is any integer and $n \ge 2$, and $A_1, \ldots, A_n$ are any sets. This is a generalized De Morgan's law.

Problem 18: Let a and b be two integers. Prove by mathematical induction that for all integers $n \ge 1$, $a^n - b^n$ is a multiple of $a - b$.

Note that you are asked to prove this by mathematical induction, so you cannot simply use the following fact:

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}).$$

Problem 19: Let x be any real number and $x > 1$. Prove that for all $n > 1$

$$(1+x)^n > 1 + nx.$$

Problem 20: Let x be any real number such that $x > 0$ and $x \neq 1$. Prove that

$$x, x^x, x^{(x^x)}, x^{(x^{(x^x)})}, \ldots$$

is an increasing sequence.

Problem 21: Prove by mathematical induction that if n is any positive odd integer, then $1 + 3^n$ is divisible by 4.

Problem 22: Prove that the sum of the cubes of any three consecutive integers is divisible by 9.

Problem 23: Prove by mathematical induction that for all $n \in \mathbf{Z}$

$$(-1)^n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

$\mathbf{Z}$ is the set of all integers.

Problem 24: Let $c(n)$ be a sequence of integers such that

$$c(0) = 1, c(1) = 1, c(2) = 3,$$

and for all $n \geq 1$,

$$c(n+2) = 3c(n+1) - 3c(n) + c(n-1).$$

Prove that for all $n \geq 0$,

$$c(n) = n^2 - n + 1.$$

Problem 25: Define $b(n)$ as follows:

$$b(n+2) + 2b(n+1) + b(n) = 0, n \geq 0,$$

and $b(0) = 1$ and $b(1) = 1$. Prove that

$$b(n) = (1 - 2n)(-1)^n, n \geq 0.$$

Problem 26: Using the same definition of $b(n)$ given in problem 25, prove that if $b(0) = 1$ and $b(1) = -3$, then

$$b(n) = (1 + 2n)(-1)^n, n \geq 0.$$

Problem 27: Let us consider a famous sequence called Fibonacci numbers:

$$\begin{aligned} &f_0 = 0, \\ &f_1 = 1, \\ &f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 2. \end{aligned}$$

Prove by mathematical induction that, for all $n \geq 0$,

$$f_n = \frac{1}{\sqrt{5}} \times \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Problem 28: Let S be a set with n elements. Prove by mathematical induction that the total number of subsets of S having exactly two elements is $n(n-1)/2$.

Problem 29: Let $p(n)$ be the maximum number of intersection points of n distinct lines in the plane. Prove by mathematical induction that for all integers $n \geq 2$, $p(n) = n(n-1)/2$.

Problem 30: Let $x_1 = 1$ and $x_{n+1} = \sqrt{x_n^2 + \frac{1}{x_n^2}}$. Prove by mathematical induction that for all $n \geq 1$,

$$1 \leq x_n \leq \sqrt{n}.$$

Problem 31: Define the harmonic numbers as

$$H_n = \sum_{1 \leq i \leq n} \frac{1}{i}, \quad n \geq 1.$$

Prove by mathematical induction that

$$H_{2^n} \geq \left(1 + \frac{n}{2}\right), \quad n \geq 0.$$

Problem 32: Let λ denote the empty string. Let A be any finite nonempty set. A *palindrome* over A can be defined as a string that reads the same forward as backward. For example, "mom" and "dad" are palindromes over the set of English alphabets.

We define a set S as follows:

1. $\lambda \in S$

2. $\forall a \in A,\ a \in S$

3. $\forall a \in A\ \forall x \in S,\ axa \in S$

4. All the elements in S must be generated by the rules above.

Prove by structural induction that S equals the set of all palindromes over A.

Problem 33: Let set S be a set of strings of a's and b's recursively defined as follows:

1. $a \in S$, $b \in S$.

2. If $\mu \in S$ and $\nu \in S$, then $\mu\nu \in S$.

3. Nothing but strings generated according to rules 1 and 2 are elements in S.

We also recursively define the reverse operation R on S as:

1. $R(a) = a$, and $R(b) = b$.

2. If $\mu \in S$, then $R(a\mu) = R(\mu)a$, and $R(b\mu) = R(\mu)b$.

Prove by structural induction that for all $\mu, \nu \in S$,

$$R(\mu\nu) = R(\nu)R(\mu).$$

Problem 34: Let S and R be as defined in the previous problem. Prove that, for all $\mu \in S$,

$$R(R(\mu)) = \mu.$$

Problem 35: Let $\Sigma = \{a, b\}$ and Λ be the null string. Define $A \subseteq \Sigma^*$ by the following rules.

1. $\Lambda \in A$.
2. If $\omega \in A$, then $a\omega b \in A$.
3. If $\mu, \nu \in A$, then $\mu\nu \in A$.
4. Every $\omega \in A$ must come from a finite number of applications of rules 1, 2, or 3.

Prove by mathematical induction that, for every $\omega \in A$, ω has equally many a's and b's.

Problem 36: Let A be defined as the previous problem. Prove by mathematical induction that, if $\omega \in A$, then the number of a's is equal to or greater than the number of b's in every prefix of ω.

Note: A prefix of ω is Λ or any initial segment of ω. For example, if $\omega = aabbad$, then Λ, a, aa, aab, $aabb$, $aabba$, and $aabbab$ are all possible prefixes of ω.

3.5 Solutions

Solution 1: The basis of the induction is not proved. It is incorrect, because for $n = 1$,

$$\sum_{1 \le k \le 1} 2^k = 2^1 = 2,$$

which is not equal to $2^{1+1} = 4$. □

Solution 2: The proof is incorrect. The proof shows that $a_0 \neq 1$ and claims that it is the basis of the induction. According to the hypothesis given in the proof, the correct basis is $a_0 \neq 1$ and $a_1 > a_0$.

To see the mistake more clearly, let's change the initial values of the sequence to $a_0 = 3$, and $a_1 = 2$. We thus have $a_2 = 1$. The inductive basis only proves that $a_0 \neq 1$, but does not mention the problem that $a_0 > a_1$. □

Solution 3:

1.
$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! \;=\; \sum_{1 \le k \le n} k \cdot k!.$$

2. In order to obtain a closed form for the sum above, we note that $k = (k+1) - 1$ for any k. Thus,

$$\begin{aligned} & 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! \\ & \quad = (2 \cdot 1! - 1!) + (3 \cdot 2! - 2!) + (4 \cdot 3! - 3!) + \cdots + ((n+1) \cdot n! - n!) \\ & \quad = (2! - 1!) + (3! - 2!) + (4! - 3!) + \cdots + ((n+1)! - n!) \\ & \quad = (n+1)! - 1. \end{aligned}$$

Therefore,

$$\sum_{1 \le k \le n} k \cdot k! = (n+1)! - 1. \tag{3.9}$$

□

Solution 4: Recall that our goal is to prove that equation (9) holds for all values of $n \geq 1$.

by mathematical induction:

- **Inductive Basis:** $n = 1$.

$$\sum_{1\leq k\leq 1} k\cdot k! = 1\cdot 1! = 1,$$
$$(1+1)! - 1 = 1.$$

That means both sides of (9) are equal to 1. **[The Basis Holds.]**

- **Inductive Hypothesis:** Suppose

$$\sum_{1\leq k\leq n} k\cdot k! = (n+1)! - 1.$$

Inductive Step:

$$\begin{aligned}\sum_{1\leq k\leq n+1} k\cdot k! &= \sum_{1\leq k\leq n} k\cdot k! + (n+1)\cdot(n+1)! \\ &= (n+1)! - 1 + (n+1)\cdot(n+1)! \qquad [\textbf{by IH}] \\ &= (n+2)\cdot(n+1)! - 1 \\ &= (n+2)! - 1.\end{aligned}$$

[The Inductive Step Holds.]

Therefore, for all $n \in \mathbf{N}$, (9) is correct. □

Solution 5:

1.

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)} = \sum_{1\leq k\leq n}\frac{1}{(3k-2)\cdot(3k+1)}.$$

2. In this problem we make use of the equality, for any k, $1 \leq k \leq n$,

$$\begin{aligned}\frac{1}{3}\left(\frac{1}{3k-2}-\frac{1}{3k+1}\right) &= \frac{1}{3}\frac{3k+1-3k+2}{(3k-2)(3k+1)} \\ &= \frac{1}{(3k-2)(3k+1)}.\end{aligned}$$

$$\begin{aligned}
&\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n-2)\cdot(3n+1)}\\
&\quad=\frac{1}{3}(\frac{1}{1}-\frac{1}{4})+\frac{1}{3}(\frac{1}{4}-\frac{1}{7})+\frac{1}{3}(\frac{1}{7}-\frac{1}{10})+\cdots+\frac{1}{3}(\frac{1}{3n-2}-\frac{1}{3n+1})\\
&\quad=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+\cdots+\frac{1}{3n-2}-\frac{1}{3n+1}\right)\\
&\quad=\frac{1}{3}\left(1-\frac{1}{3n+1}\right).
\end{aligned}$$

Therefore, the closed form expression of the sum is

$$\sum_{1\le k\le n}\frac{1}{(3k-2)\cdot(3k+1)} \;=\; \frac{1}{3}\left(1-\frac{1}{3n+1}\right). \tag{3.10}$$

□

Solution 6: Our goal is to prove (10) by mathematical induction.

- **Inductive Basis:** $n = 1$.

$$\begin{aligned}
\text{LHS} &= \frac{1}{3\cdot 1-2}\times\frac{1}{3\cdot 1+1} \;=\; \frac{1}{4},\\
\text{RHS} &= \frac{1}{3}\left(1-\frac{1}{3\cdot 1+1}\right) \;=\; \frac{1}{4}.
\end{aligned}$$

[The Basis Holds.]

- **Inductive Hypothesis:** Suppose

$$\sum_{1\le k\le n}\frac{1}{(3k-2)\cdot(3k+1)}=\frac{1}{3}\left(1-\frac{1}{3n+1}\right).$$

Inductive Step:

$$\begin{aligned}
&\sum_{1\le k\le n+1} \frac{1}{(3k-2)\cdot(3k+1)}\\
&= \left(\sum_{1\le k\le n} \frac{1}{(3k-2)\cdot(3k+1)}\right) + \frac{1}{(3n+1)\cdot(3n+4)}\\
&= \frac{1}{3}\left(1-\frac{1}{3n+1}\right) + \frac{1}{(3n+1)\cdot(3n+4)} && \textbf{[by IH]}\\
&= \frac{1}{3}\left(1-\frac{1}{3n+1}+\frac{3}{(3n+1)\cdot(3n+4)}\right)\\
&= \frac{1}{3}\left(1-\frac{3n+4-3}{(3n+1)\cdot(3n+4)}\right)\\
&= \frac{1}{3}\left(1-\frac{3n+1}{(3n+1)\cdot(3n+4)}\right)\\
&= \frac{1}{3}\left(1-\frac{1}{3n+4}\right)
\end{aligned}$$

[The Inductive Step Holds.]

Therefore, for all $n \in \mathbf{N}$, (10) is correct. □

Solution 7: Let $D_n = \mathbf{N}$, and [5]

$$P(n)^6 : \sum_{1\le k\le n} k(k+1) = \frac{1}{3}n(n+1)(n+2).$$

- **Inductive Basis:** $n = 1$.

$$1 \times 2 = \frac{1}{3}(1 \times 2 \times 3).$$

Therefore, $P(1) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $P(n) = T$, i.e.

$$\sum_{1\le k\le n} k(k+1) = \frac{1}{3}n(n+1)(n+2).$$

[5] Beginning from this problem and onwards, we will explicitly define the predicate and its domain.

[6] **Note:** $P(n)$ is a predicate; for any fixed value of n, its value is either $True$ or $False$. It does not represent the sum or its value on the other side of the equality, i.e.,

$$P(n) \ne \sum_{1\le k\le n} k(k+1) \text{ and } P(n) \ne \frac{1}{3}n(n+1)(n+2).$$

- **Inductive Step:** We want to prove that $P(n+1) = True$. In other words, we want to prove that the following equality is correct:

$$\sum_{1\le k\le n+1} k(k+1) = \frac{1}{3}(n+1)(n+2)(n+3).$$

$$\begin{aligned}
\sum_{1\le k\le n+1} k(k+1) &= \left(\sum_{1\le k\le n} k(k+1)\right) + (n+1)((n+1)+1) \\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \qquad \textbf{[by IH]} \\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\
&= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\
&= \frac{(n+1)(n+2)(n+3)}{3} \\
&= \frac{(n+1)((n+1)+1)((n+1)+2)}{3} \\
&= \frac{1}{3}(n+1)(n+2)(n+3).
\end{aligned}$$

$P(n+1) = T.$ **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n)$ is true. □

Solution 8: Let $D_n = \mathbf{N}$, and

$$P(n) : \sum_{0\le k\le n} 3^k = \frac{(3^{n+1}-1)}{2}.$$

- **Inductive Basis:** $n = 1$.

$$\sum_{0\le k\le 1} 3^k = 3^0 + 3^1 = 4,$$

$$\frac{(3^{1+1}-1)}{2} = \frac{9-1}{2} = 4.$$

Therefore, $P(1) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $P(n) = T$, i.e.,

$$\sum_{0\le k\le n} 3^k = \frac{(3^{n+1}-1)}{2}.$$

- **Inductive Step:** We want to prove that $P(n+1)$ is true, i.e., we want to prove that

$$\sum_{0\le k\le n+1} 3^k = \frac{(3^{n+2}-1)}{2}.$$

$$\begin{aligned}
\sum_{0\le k\le n+1} 3^k &= \left(\sum_{0\le k\le n} 3^k\right) + 3^{n+1} \\
&= \frac{3^{n+1}-1}{2} + 3^{n+1} \qquad \textbf{[by IH]} \\
&= \frac{3^{n+1}-1+2\cdot 3^{n+1}}{2} \\
&= \frac{3\cdot 3^{n+1}-1}{2} \\
&= \frac{3^{n+1+1}-1}{2} \\
&= \frac{3^{n+2}-1}{2}.
\end{aligned}$$

$P(n+1)=T$. **[The Inductive Step Holds.]**

Therefore, $P(n)$ is true for all n in D_n. □

Solution 9: Let $D_n = \mathbf{N}$, and

$$P(n): \sum_{1\le k\le n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

- **Inductive Basis:** $n=1$.

$$1^2 = \frac{1}{6}\times 1\times 2\times 3$$

Therefore, $P(1)$ is true. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n$, and assume $P(i)=T$, i.e.,

$$\sum_{1\le k\le i} k^2 = \frac{1}{6}i(i+1)(2i+1).$$

- **Inductive Step:** Consider $P(i+1)$,

$$\begin{aligned}\sum_{1\le k\le i+1} k^2 &= \left(1^2+2^2+\cdots+i^2\right)+(i+1)^2\\ &= \left(\sum_{1\le k\le i} k^2\right)+(i+1)^2\\ &= \frac{1}{6}i(i+1)(2i+1)+(i+1)^2 \qquad \textbf{[by IH]}\\ &= \frac{1}{6}(i+1)(i(2i+1)+6(i+1))\\ &= \frac{1}{6}(i+1)(2i^2+7i+6)\\ &= \frac{1}{6}(i+1)(i+2)(2i+3))\\ &= \frac{1}{6}(i+1)((i+1)+1)(2(i+1)+1)).\end{aligned}$$

$P(i+1)=T$. **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n)$ is $True$. □

Solution 10: Let $D_n = \mathbf{N}$, and

$$P(n): \sum_{1\le k\le n} k(k-1) = \frac{1}{3}(n+1)n(n-1).$$

- **Inductive Basis:** $n=1$.

$$1\times(1-1)=\frac{1}{3}\times(1+1)\times 1\times(1-1).$$

Therefore, $P(1)=True$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i\in D_n$, and assume $P(i)=T$, i.e.,

$$\sum_{1\le k\le i} k(k-1)=\frac{1}{3}(i+1)i(i-1).$$

- **Inductive Step:** Consider $P(i+1)$,

$$\begin{aligned}\sum_{1\le k\le i+1} k(k-1) &= \left(\sum_{1\le k\le i} k(k-1)\right) + (i+1)((i+1)-1)\\ &= \frac{1}{3}(i+1)i(i-1) + i(i+1) \qquad \textbf{[by IH]}\\ &= \frac{1}{3}(i+1)i(i-1+3)\\ &= \frac{1}{3}(i+1)i(i+2).\end{aligned}$$

$P(i+1) = T$. **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n)$ is $True$. □

Solution 11: Let $D_n = \mathbf{N}$, and

$$P(n) \ : \ \sum_{1\le k\le n} k(k-1)(k-2) = \frac{1}{4}(n+1)n(n-1)(n-2).$$

- **Inductive Basis:** For $n = 1$, the LHS of $P(n)$ is $1(1-1)(1-2) = 0$, and the RHS is $\frac{1}{4}(1+1)1(1-1)(1-2) = 0$. Since both sides are equal, $P(1) = (0 = 0) = T$. **[The Basis Holds.]**
- **Inductive Hypothesis:** Let $i \in D_n$, and assume $P(i) = T$, i.e.,

$$\sum_{1\le k\le i} k(k-1)(k-2) = \frac{1}{4}(i+1)i(i-1)(i-2).$$

- **Inductive Step:** To prove that $P(i+1)$ is true, we need to show that

$$\sum_{1\le k\le i+1} k(k-1)(k-2) = \frac{1}{4}(i+2)(i+1)i(i-1).$$

$$\begin{aligned}&\sum_{1\le k\le i+1} k(k-1)(k-2)\\ &= \sum_{1\le k\le i} k(k-1)(k-2) + (i+1)i(i-1)\\ &= \frac{1}{4}(i+1)i(i-1)(i-2) + (i+1)i(i-1) \qquad \textbf{[by IH]}\\ &= \frac{1}{4}(i+1)i(i-1)\left((i-2)+4\right)\\ &= \frac{1}{4}(i+1)i(i-1)(i+2).\end{aligned}$$

$P(i+1) = T.$ **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n)$ is $True$. □

Solution 12: Let's observe the following properties first. Let $f(k)$ and $g(k)$ be two functions of k, and a be any constant. We have the following equalities.

$$\begin{aligned}\sum_{1\le k\le n}(f(k)+g(k)) &= \sum_{1\le k\le n} f(k) + \sum_{1\le k\le n} g(k).\\ \sum_{1\le k\le n} af(k) &= a\left(\sum_{1\le k\le n} f(k)\right).\end{aligned}$$

One can verify that,

$$\begin{aligned}k^3 &= k(k-1)(k-2)+3k^2-2k.\\ &= k(k-1)(k-2)+2k(k-1)+k^2.\end{aligned}$$

Therefore,

$$\sum_{1\le k\le n} k^3 = \left(\sum_{1\le k\le n} k(k-1)(k-2)\right) + 2\left(\sum_{1\le k\le n} k(k-1)\right) + \sum_{1\le k\le n} k^2.$$

Then, we use the results from the previous problems to get,

$$\begin{aligned}\sum_{1\le k\le n} k^3 &= \frac{1}{4}(n+1)n(n-1)(n-2)+\frac{2}{3}(n+1)n(n-1)+\frac{1}{6}n(n+1)(2n+1)\\ &= \frac{1}{12}n(n+1)[3(n-1)(n-2)+8(n-1)+2(2n+1)]\\ &= \frac{1}{12}n(n+1)(3n^2-9n+6+8n-8+4n+2)\\ &= \frac{1}{12}n(n+1)(3n+3n^2)\\ &= \frac{1}{4}n^2(n+1)^2.\end{aligned}$$

□

Solution 13: Let $D_n = \mathbf{N}$, and

$$P(n): \quad \sum_{1\le k\le n} k^3 = \frac{1}{4}n^2(n+1)^2.$$

- **Inductive Basis:** $n = 1$. It is easy to verify that, for $n = 1$, both sides of the equality are 1i, i.e.,

$$1^3 = \frac{1}{4}1^2(1+1)^2.$$

Therefore, $P(1) = True$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n$, and assume $P(i) = T$, i.e.,

$$\sum_{1 \le k \le i} k^3 = \frac{1}{4}i^2(i+1)^2.$$

- **Inductive Step:** Consider $P(i+1)$,

$$\begin{aligned}
\sum_{1 \le k \le i+1} k^3 &= \sum_{1 \le k \le i} k^3 + (i+1)^3 \\
&= \frac{1}{4}i^2(i+1)^2 + (i+1)^3 \qquad [\textbf{by IH}] \\
&= \frac{1}{4}(i+1)^2(i^2 + 4(i+1)) \\
&= \frac{1}{4}(i+1)^2(i^2 + 4i + 4) \\
&= \frac{1}{4}(i+1)^2(i+2)^2.
\end{aligned}$$

Thus, $P(i+1) = T$. **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n) = T$. □

Solution 14: Let $D_n = \{1, 3, 5, 7, 9, \ldots\}$, and define,

$$P(n) : \sum_{0 \le k \le n} (-2)^k = \frac{1}{3}(1 - 2^{n+1}).$$

- **Inductive Basis:** $n = 1$. We choose 1 as the basis because 1 is the first element of D_n in our enumeration, and note that the LHS of the equality is

$$\sum_{0 \le k \le 1} (-2)^k = (-2)^0 + (-2)^1 = -1,$$

and the RHS of the equality is

$$\frac{1}{3}(1 - 2^2) = -1.$$

Therefore, $P(1) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n$, and assume $P(i) = T$, i.e.,

$$\sum_{0 \le k \le i} (-2)^k = \frac{1}{3}(1 - 2^{i+1}).$$

- **Inductive Step:** Let $n = i + 2$. We choose $i + 2$ because $i + 2$ is the positive odd integer next to i.

$$\begin{aligned}
\sum_{0 \le k \le i+2} (-2)^k &= \sum_{0 \le k \le i} (-2)^k + \sum_{i+1 \le k \le i+2} (-2)^k \\
&= \frac{1}{3}(1 - 2^{i+1}) + (-2)^{i+1} + (-2)^{i+2} \qquad [\textbf{by IH}] \\
&= \frac{1}{3}[1 - 2^{i+1} + 3(-2)^{i+1} + 3(-2)^{i+2}] \\
&= \frac{1}{3}[1 - 2^{i+1} + 3(-2)^{i+1} + 3(-2)(-2)^{i+1}] \\
&= \frac{1}{3}[1 - 2^{i+1} + 3(-2)^{i+1} - 6(-2)^{i+1}] \\
&= \frac{1}{3}[1 - (1 - 3 + 6) \times 2^{i+1}] \\
&= \frac{1}{3}(1 - (-2)^2(-2)^{i+1}) \quad \text{since } i+1 \text{ is even} \\
&= \frac{1}{3}(1 - (-2)^{i+3}) \\
&= \frac{1}{3}(1 - 2^{i+3}) \quad \text{since } i+3 \text{ is even.}
\end{aligned}$$

$P(i + 2) = T$. **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n)$ is true. □

Solution 15: Define $P(n)$ for all $n \ge 1$ as,

$$P(n) : \sum_{1 \le k \le n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

- **Inductive Basis:** For $n = 1$, it is easy to verify that both sides of the equality are equal to $\frac{1}{2}$.

$$\text{LHS} = \sum_{1 \le k \le 1} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} = 1/2.$$

$$\text{RHS} = 1 - \frac{1}{1+1} = 1/2.$$

Therefore, $P(1)$ is true. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $P(n)$ is true, i.e.,

$$\sum_{1\le k\le n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}.$$

- **Inductive Step:** To prove $P(n+1)$ is true.

$$\begin{aligned}\sum_{1\le k\le n+1} \frac{1}{k(k+1)} &= \sum_{1\le k\le n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)} \\ &= 1 - \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} \qquad \textbf{[by IH]} \\ &= 1 - \frac{(n+2)-1}{(n+1)(n+2)} \\ &= 1 - \frac{n+1}{(n+1)(n+2)} \\ &= 1 - \frac{1}{n+2}\end{aligned}$$

Therefore, $P(n+1)$ is true. **[The Inductive Step Holds.]**

□

Solution 16: Define $P(n)$ for all $n \ge 1$ as,

$$P(n) : \sum_{1\le k\le n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

- **Inductive Basis:** $n = 1$.

$$\sum_{1\le k\le 1} \frac{2k+1}{k^2(k+1)^2} = \frac{2+1}{1^2 \cdot 2^2} = 3/4.$$

$$1 - \frac{1}{(1+1)^2} = 3/4.$$

Therefore, $P(1)$ is true. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $P(n)$ is true, i.e.

$$\sum_{1\le k\le n} \frac{2k+1}{k^2(k+1)^2} = 1 - \frac{1}{(n+1)^2}.$$

- **Inductive Step:** To prove $P(n+1)$ is true.

$$\begin{aligned}
\sum_{1\le k\le n+1} \frac{2k+1}{k^2(k+1)^2} &= \sum_{1\le k\le n} \frac{2k+1}{k^2(k+1)^2} + \frac{2(n+1)+1}{(n+1)^2(n+2)^2} \\
&= 1 - \frac{1}{(n+1)^2} + \frac{2(n+1)+1}{(n+1)^2(n+2)^2} \qquad \textbf{[by IH]} \\
&= 1 - \frac{(n+2)^2 - 2(n+1) - 1}{(n+1)^2(n+2)^2} \\
&= 1 - \frac{n^2+4n+4-2n-2-1}{(n+1)^2(n+2)^2} \\
&= 1 - \frac{n^2+2n+1}{(n+1)^2(n+2)^2} \\
&= 1 - \frac{(n+1)^2}{(n+1)^2(n+2)^2} \\
&= 1 - \frac{1}{(n+2)^2}
\end{aligned}$$

Therefore, $P(n+1)$ is true. **[The Inductive Step Holds.]**

□

Solution 17: Let A_i denote a set for any $i \in \mathbf{N}$, and $P(n)$ be the predicate, for $n \in \{2, 3, 4, \ldots\}$,

$$P(n): \overline{\bigcup_{1\le i\le n} A_i} = \bigcap_{1\le i\le n} \overline{A_i}.$$

- **Inductive Basis:** For $n = 2$, we need to prove that

$$\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}.$$

It is the classical De Morgan's law in logic, and is known to be true. Hence, the basis follows.

- **Inductive Hypothesis:** Assume $n = k$ and $P(k) = T$, i.e.,

$$\overline{\bigcup_{1\le i\le k} A_i} = \bigcap_{1\le i\le k} \overline{A_i}.$$

- **Inductive Step:** Let $n = k+1$.

$$\overline{\bigcup_{1\le i\le k+1} A_i} = \overline{(\bigcup_{1\le i\le k} A_i) \cup A_{k+1}}.$$

If we take $(\bigcup_{1\le i\le k} A_i)$ as A, and A_{k+1} as B, we have

$$\overline{(\bigcup_{1\le i\le k} A_i) \cup A_{k+1}} = \overline{(\bigcup_{1\le i\le k} A_i)} \cap \overline{A_{k+1}}.$$

Using the inductive hypothesis, we have

$$\begin{aligned}\overline{(\bigcup_{1\le i\le k} A_i)} \cap \overline{A_{k+1}} &= (\bigcap_{1\le i\le k} \overline{A_i}) \cap \overline{A_{k+1}} \\ &= \bigcap_{1\le i\le k+1} \overline{A_i}.\end{aligned}$$

That proves $P(k+1) = T$. **[The Inductive Step Holds.]**

Therefore, $\forall n \ge 2, P(n)$ is T.

Note: If we want to claim that $[\forall n \ge 1, P(n)]$ is also true, we have to prove the special case, $n = 1$, and the inductive basis, $n = 2$, separately. It is trivial to prove $P(1)$ is true, but $P(1) = T$ is not the basis of the induction. We have to prove $P(2) = T$ as the inductive basis, because $P(1)$ does not imply $P(2)$.

$\square$

Solution 18: Let $D_n = \mathbf{N}$, and

$$P(n) : (a-b)|(a^n - b^n),$$

where $(a-b)|(a^n-b^n)$ means a^n-b^n is divisible by $a-b$. In other words, we have to prove that for any fixed n, there exists an integer k such that $a^n-b^n = a(a-b)$. Note that k may be a function of a, b, and n.

- **Inductive Basis:** For $n = 1$, $a^n - b^n = a - b$ and it is obvious that $(a-b)$ divides $(a-b)$, i.e.,

$$(a-b)|(a^1 - b^1) \text{ is true.}$$

Therefore, $P(1) = True$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n, i > 1$, and assume $P(i) = T$, i.e.,

$$(a-b)|(a^i - b^i),$$

- **Inductive Step:** We want to know if $a^{i+i} - b^{i+1}$ is divisible by $a - b$.

$$a^{i+1} - b^{i+1} = aa^i - ba^i + ba^i - bb^i = a^i(a - b) + b(a^i - b^i).$$

From the inductive hypothesis, we know that there exists an integer k such that,

$$(a^i - b^i) = k(a - b).$$

Therefore,

$$\begin{aligned} a^{i+1} - b^{i+1} &= a^i(a - b) + bk(a - b) \\ &= (a - b)(a^i + bk). \end{aligned}$$

It is clear that $a^i + bk$ is an integer because a, b, i, and k are all integers. Thus,

$$(a - b)|(a^{i+1} - b^{i+1}).$$

$P(i + 1)$ is true. **[The Inductive Step Holds.]**

Therefore, for all $n \geq 1$, $a^n - b^n$ is divisible by $a - b$. □

Solution 19: Suppose $x > 0, D_n = \{2, 3, 4, \ldots\}$, and

$$P(n) : (1 + x)^n > 1 + nx.$$

- **Inductive Basis:** $n = 2$.

$$(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x, \quad \text{because } n^2 > 0.$$

[The Basis Holds.]

- **Inductive Hypothesis:** Let $i \in D_n$, and assume $P(i) = T$, i.e.,

$$(1 + x)^i > 1 + ix.$$

- **Inductive Step:** We want to prove that $P(i + 1)$ is true.

$$\begin{aligned} (1 + x)^{i+1} &= (1 + x)^i(1 + x) \\ &> (1 + ix)(1 + x) \qquad [\textbf{by IH}] \text{ and } x > 0 \\ &= 1 + (i + 1)x + ix^2 \\ &> 1 + (i + 1)x \text{ by } n^2 > 0, i > 0. \end{aligned}$$

[The Inductive Step Holds.]

Note: The given condition $x > 0$ and the inductive hypothesis are both needed to claim that

$$(1+x)^i(1+x) > (1+ix)(1+x).$$

One can easily find a counter example if $x < 0$.

□

Solution 20: We express the sequence

$$x, x^x, x^{(x^x)}, x^{(x^{(x^x)})}, \ldots$$

as

$$a_1, a_2, a_3, \ldots, \ldots$$

. Thus,

$$\begin{aligned} a_1 &= x, \\ a_2 &= x^x = x^{a_1}, \\ a_3 &= x^{(x^x)} = x^{a_2}, \\ &\cdots \; \cdots \end{aligned}$$

In general, $a_n = x^{a_{n-1}}$ for $n \geq 2$. Let $D_n = \mathbf{N}$, and

$$P(n) : a_n < a_{n+1}.$$

We want to prove that, if $x > 1$, then for all n, $P(n)$ is true.

- **Inductive Basis:** $n = 1$. We discuss the two cases: $x > 1$ and $0 < x < 1$.

  1. Suppose that $0 < x < 1$. In this case, we know that $\log x < 0$. Since $0 < x < 1$, we have $\log x < x \cdot \log x$. It follows that $x < x^x$.
  2. If $x > 1$, then it is clear that $x < x^x$ if $x > 1$.

  Thus, in both cases, we have

  $$a_1 < a_2, \quad P(1) = True.$$

  [The Basis Holds.]

- **Inductive Hypothesis:** Assume $P(i)$ is true for some i in D_n, i.e.

  $$a_i < a_{i+1}.$$

- **Inductive Step:** We want to prove $P(i+1)$ is also true. Using the hypothesis, we have

$$\begin{aligned} a_i < a_{i+1} &\Rightarrow x^{a_i} < x^{a_{i+1}} \qquad \text{because } x > 1 \\ &\Rightarrow P(i+1) = T. \end{aligned}$$

[The Inductive Step Holds.]

Therefore, it is an increasing sequence. □

Solution 21: Let D_n be the set of all positive odd integers, i.e, $D_n = \{1, 3, 5 \ldots\}$, and define

$$P(n) : 1 + 3^n \text{ is divisible by } 4.$$

- **Inductive Basis:** $n = 1$. We choose 1 as the base because 1 is the first element in D_n. Apparently, $1 + 3^1 = 4$, which is divisible by 4. Therefore, $P(1) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $P(n) = T$, i.e., $1 + 3^n$ is divisible by 4. Note: n is a positive odd integer.

- **Inductive Step:** We want to prove $P(n+2) = T$, i.e., we want to prove that $1 + 3^{n+2}$ is divisible by 4.

  Note: We choose $n+2$ instead of $n+1$ because $n+2$ is the odd number next to n fixed in the inductive hypothesis, whereas $n+1$ is not an element in D_n.

  From the inductive hypothesis, we can assume that $1 + 3^n = 4k$, where k is an integer. We have

$$\begin{aligned} 1 + 3^{n+2} &= 1 + 9 \cdot 3^n \\ &= (1 + 3^n) + 8 \cdot 3^n \\ &= 4k + 8 \cdot 3^n \\ &= 4(k + 2 \cdot 3^n). \end{aligned}$$

  Because k and n are integers, $k + 2 \cdot 3^n$ must be an integer. Therefore, $1 + 3^{n+2}$ is divisible by 4, and hence $P(n+2) = T$.

[The Inductive Step Holds.]

Therefore, $P(n)$ is true for all n in D_n. □

Solution 22: The domain of this problem includes both positive and negative integers. The easiest way is to split the domain into two parts: positive integers and negative integers. Then, we prove the statement by mathematical induction in the two sub-domain separately.

(1) Let $D_n = \{0, 1, 2, 3, \ldots\}$ Define:

$$P(n) : 9 \,\big|\, \left(n^3 + (n+1)^3 + (n+2)^3\right) .$$

- **Inductive Basis:** $n = 0$.

$$0^3 + 1^3 + 2^3 = 9.$$

$P(0)$ is true. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $i \in D_n$, and $P(i)$ is true, i.e.,

$$i^3 + (i+1)^3 + (i+2)^3 = 9k,$$

for some integer k.

- **Inductive Step:** Prove $P(i+1)$ is true.

$$\begin{aligned}
&(i+1)^3 + (i+2)^3 + (i+3)^3 \\
&\quad = (i+1)^3 + (i+2)^3 + i^3 + 9i^2 + 27i + 27 \\
&\quad = (i^3 + (i+1)^3 + (i+2)^3) + 9i^2 + 27i + 27 \\
&\quad = 9k + 9(i^2 + 3i + 3) \qquad \textbf{[by IH]} \\
&\quad = 9(k + i^2 + 3i + 3)
\end{aligned}$$

Because $k + i^2 + 3i + 3$ is an integer, therefore

$$9 \,\big|\, \left((i+1)^3 + (i+2)^3 + (i+3)^3\right) ,$$

$P(i+1)$ is true. **[The Inductive Step Holds.]**

(2) Let $D_n = \{0, -1, -2, -3, \ldots\}$ Define:

$$P(n) : 9 \,\big|\, \left(n^3 + (n-1)^3 + (n-2)^3\right)$$

- **Inductive Basis:** $n = 0$.

$$0^3 + 1^3 + 2^3 = 9.$$

$P(0)$ is true. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $i \in D_n$, and $P(i)$ is true, i.e.,

$$i^3 + (i-1)^3 + (i-2)^3 = 9k,$$

for some integer k.

- **Inductive Step:** Prove $P(i-1)$ is true.

$$\begin{aligned}&(i-1)^3 + (i-2)^3 + (i-3)^3 \\ &\quad = (i-1)^3 + (i-2)^3 + i^3 - 9i^2 + 27i - 27 \\ &\quad = (i^3 + (i-1)^3 + (i-2)^3) - 9i^2 + 27i - 27 \\ &\quad = 9k + 9(-i^2 + 3i - 3) \qquad [\textbf{by IH}] \\ &\quad = 9(k - i^2 + 3i - 3)\end{aligned}$$

Because $k - i^2 + 3i - 3$ is an integer, therefore

$$9 \,\big|\, \left((i-1)^3 + (i-2)^3 + (i-3)^3\right),$$

$P(i-1)$ is true. **[The Inductive Step Holds.]**

Putting the two domains together, we have that the sum of the cubes of any three consecutive integers is divisible by 9. □

Solution 23: We first prove that the statement is correct in the non-negative part of the domain. Let $D_n = \{0\} \cup \mathbf{N}$, and define

$$P(n):\ (-1)^n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ -1 & \text{if } n \text{ is odd.} \end{cases}$$

We will prove that $P(n) = T$ for all $n \in D_n$.

- **Inductive Basis:** $n = 0$.

  0 is even and $(-1)^0 = 1$. Thus $P(0) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** $n = i$.

  Suppose $P(i) = T$, i.e.,

$$(-1)^i = \begin{cases} 1 & \text{if } i \text{ is even,} \\ -1 & \text{if } i \text{ is odd.} \end{cases}$$

- **Inductive Step:** $n = i + 1$.

$$\begin{aligned}(-1)^{i+1} &= -1 \cdot (-1)^i \\ &= \begin{cases} -1 \cdot 1 & \text{if } i \text{ is even,} \\ -1 \cdot -1 & \text{if } i \text{ is odd.} \end{cases} \\ &= \begin{cases} -1 & \text{if } i \text{ is even,} \\ 1 & \text{if } i \text{ is odd.} \end{cases} \\ &= \begin{cases} -1 & \text{if } i+1 \text{ is odd,} \\ 1 & \text{if } i+1 \text{ is even.} \end{cases}\end{aligned}$$

Therefore, $P(i+1)$ is true. **[The Inductive Step Holds.]**

We have proved that the predicate is true for all non-negative integers. We can use the same technique shown in the previous problem, i.e., we can further prove that the statement is correct for the other part of the domain (negative integers). Or, we can use the result we just got and the following arguments.

From the proved result above, if n is a non-negative integer, we know that $2n$ is even and $(-1)^{2n} = 1$. Observe the following fact: if $n \in \mathbf{N}$, then

$$(-1)^{-n} = \frac{1}{(-1)^n} = \frac{(-1)^{2n}}{(-1)^n} = (-1)^n.$$

Therefore, we can claim that for all $i \in \mathbf{Z}$, the predicate $P(i)$ is always true. That proves the theorem. □

The following is another proof for the above problem. It is a bit awkward, but it shows how to reorder the domain so we can examine the entire domain without missing any element of it. Let's consider the following ordered sequence,

$$\mathcal{Z} \;:\; 0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$$

It is clear that any number in $\mathbf{Z}$ must be in the sequence $\mathcal{Z}$ somewhere. Let z_i denote the i^{th} number in $\mathcal{Z}$, where $i \geq 1$. Let $D_n = \mathbf{N}$. Define

$$P(n): \; (-1)^{z_n} = \begin{cases} 1 & \text{if } z_n \text{ is even,} \\ -1 & \text{if } z_n \text{ is odd.} \end{cases}$$

Thus, our original problem can be viewed as:

Prove that for all $n \in D_n$, $P(n)$ is true.

- **Inductive Basis:** $n = 1$.

  $z_1 = 0$, therefore z_1 is even and $(-1)^0 = 1$. Thus $P(1) = T$.

  [The Basis Holds.]

- **Inductive Hypothesis:** $n = i$.

  Suppose $P(i) = T$, i.e.

$$(-1)^{z_i} = \begin{cases} 1 & \text{if } z_i \text{ is even,} \\ -1 & \text{if } z_i \text{ is odd.} \end{cases}$$

- **Inductive Step:** $n = i + 1$.

  There are only two possible values for z_{i+1} in terms of z_i, i.e., .$z_{i+1} = -z_i$ if z_i is positive or $z_{i+1} = -z_i + 1$ if z_i is negative. We will discuss this by cases.

  case 1: $z_{i+1} = -z_i$.

$$(-1)^{z_{i+1}} = (-1)^{-z_i} = \frac{1}{(-1)^{z_i}} = \begin{cases} 1/1 \quad = 1 & \text{if } z_i \text{ is even,} \\ 1/-1 \quad = -1 & \text{if } z_i \text{ is odd.} \end{cases}$$

  From $\mathcal{Z}$ we know that in this case if z_i is even, then z_{i+1} is even, and if z_i is odd, then z_{i+1} is odd. Therefore,

$$(-1)^{z_{i+1}} = \begin{cases} 1 & \text{if } z_{i+1} \text{ is even,} \\ -1 & \text{if } z_{i+1} \text{ is odd.} \end{cases}$$

  case 2: $z_{i+1} = -z_i + 1$.

$$(-1)^{z_{i+1}} = (-1)^{-z_i+1} = \frac{-1}{(-1)^{z_i}} = \begin{cases} -1/1 = -1 & \text{if } z_i \text{ is even,} \\ -1/-1 = 1 & \text{if } z_i \text{ is odd.} \end{cases}$$

  From $\mathcal{Z}$ we know that in this case if z_i is even, then z_{i+1} is odd, and if z_i is odd, then z_{i+1} is even. Therefore,

$$(-1)^{z_{i+1}} = \begin{cases} 1 & \text{if } z_{i+1} \text{ is even,} \\ -1 & \text{if } z_{i+1} \text{ is odd.} \end{cases}$$

  In both cases, $P(i + 1)$ is true. **[The Inductive Step Holds.]**

This proves the problem. □

Solution 24: Let us rewrite the recurrent relation first:

$$\begin{aligned} \forall n \geq 1,\ c(n+2) &= 3c(n+1) - 3c(n) + c(n-1) \\ \Rightarrow \forall n \geq 3,\ c(n) &= 3c(n-1) - 3c(n-2) + c(n-3). \end{aligned}$$

Given $c(0) = c(1) = 1$, and $c(2) = 3$.

To prove that for all $n \geq 0$, $c(n) = n^2 - n + 1$, let $D_n = \mathbf{N}^0$, and define

$$P(n) \ :\ c(n) = n^2 - n + 1.$$

- **Inductive Basis:** $n = 0, 1, 2$.

$$\begin{aligned} c(0) &= 0 - 0 + 1 = 1. \\ c(1) &= 1 - 1 + 1 = 1. \\ c(2) &= 4 - 2 + 1 = 3. \end{aligned}$$

Therefore, $P(0) = P(1) = P(2) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume that, for a fixed $n \in D_n$, if $0 \le i \le n$, then $P(i) = T$, i.e.,

$$c(i) = i^2 - i + 1.$$

This is a strong hypothesis.

- **Inductive Step:** To prove that $P(n+1) = T$.

To calculate the value of $c(n+1)$, we can use the values of $c(n)$, $c(n-1)$, and $c(n-2)$ given in the strong inductive hypothesis, because the arguments, n, $n-1$, and $n-2$, are in the domain of the inductive hypothesis. We have:

$$\begin{aligned} c(n) &= n^2 - n + 1, \\ c(n-1) &= (n-1)^2 - (n-1) + 1, \\ c(n-2) &= (n-2)^2 - (n-2) + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} &c(n+1) \\ &= 3c(n) - 3c(n-1) + c(n-2) \\ &= 3(n^2 - n + 1) - 3((n-1)^2 - (n-1) + 1) + (n-2)^2 - (n-2) + 1 \\ &= 3n^2 - 3n + 3 - 3n^2 + 9n - 9 + n^2 - 5n + 7 \\ &= n^2 + n + 1 \\ &= (n^2 + 2n + 1) - (n+1) + 1 \\ &= (n+1)^2 - (n+1) + 1. \end{aligned}$$

We have the result above by using the definition of $c(n+1)$ and the hypothesis, and the result agrees with the given closed-form formula for $c(n+1)$. **[The Inductive Step Holds.]**

This completes the proof. □

Solution 25: Please be careful to adjust the domain of n as shown in the following.

$$\begin{aligned} &\forall n \geq 0[b(n+2) + 2b(n+1) + b(n) = 0] \\ \equiv\ &\forall n \geq 0[b(n+2) = -2b(n+1) - b(n)] \\ \equiv\ &\forall n \geq 2[b(n) = -2b(n-1) - b(n-2)]. \end{aligned}$$

That's why this problem has to state that $b(n)$ is defined by the equation after first two values. Please compare this problem to the previous one[7].

Define $b(0) = b(1) = 1$. Let $D_n = \mathbf{N}^0$ and

$$P(n) : b(n) = (1 - 2n)(-1)^n.$$

- **Inductive Basis:** $n = 0$, and $n = 1$.

  We have to take $P(0)$ and $P(1)$ as our basis, because both $b(-1)$ and $b(-2)$ are undefined, and hence $b(0)$ and $b(1)$ cannot be obtained by using the equation $b(n) = -2b(n-1) - b(n-2)$ in the inductive step. [Same reason for the next problem.]

  $$b(0) = 1 = (1 - 0)(-1)^0.$$
  $$b(1) = 1 = (1 - 2)(-1)^1.$$

  $P(0) = P(1) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n$, and $1 \leq i$.
  Suppose, if $n \leq i$, then $P(n) = T$, i.e.,

  $$\forall n \in D_n, (n \leq i) \Rightarrow (b(n) = (1 - 2n)(-1)^n).$$

- **Inductive Step:** Let $n = i + 1$.

$$\begin{aligned} b(n) &= b(i+1) \\ &= -2b((i+1) - 1) - b((i+1) - 2) \\ &= -2b(i) - b(i-1) \\ &= -2(1 - 2i)(-1)^i - (1 - 2(i-1))(-1)^{i-1} \\ &= -2(1 - 2i)(-1)^i - (3 - 2i)(-1)^{i-1} \\ &= (-1)^{i-1}(-2(1 - 2i)(-1) - (3 - 2i)) \\ &= (-1)^{i-1}(-1 - 2i) \\ &= (-1)^2(-1)^{i-1}(1 - 2 - 2i) \\ &= (1 - 2(i+1))(-1)^{i+1} \\ &= (1 - 2n)(-1)^n. \end{aligned}$$

[7] One may ask why we don't define $b(n)$ as

$$\forall n \geq 0[b(n) = -b(n+2) - 2b(n+1)]$$

by using the given equation directly without adjusting the domain of n. Can we define a function based on its future values instead of its previous values? Theoretically, the answer is yes, but that's beyond the scope of our interest of using mathematical induction.

$P(i+1) = T.$ **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n)$ is true. □

Solution 26: Define $b(0) = 1, b(1) = -3$. Let $D_n = \mathbf{N}^0$, and

$$P(n) : b(n) = (1 + 2n)(-1)^n.$$

- **Inductive Basis:** $n = 0$, and $n = 1$.

$$b(0) = 1 = (1 + 0)(-1)^0.$$
$$b(1) = -3 = (1 + 2)(-1)^1.$$

$P(0) = P(1) = T.$ **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n$, and $1 \leq i$.
Suppose, if $n \leq i$, then $P(n) = T$, i.e.

$$\forall n \in D_n, (n \leq i) \Rightarrow (b(n) = (1 + 2n)(-1)^n).$$

- **Inductive Step:** Let $n = i + 1$.

$$\begin{aligned}
b(n) &= b(i+1) \\
&= -2b((i+1)-1) - b((i+1)-2) \\
&= -2b(i) - b(i-1) \\
&= -2(1+2i)(-1)^i - (1+2(i-1))(-1)^{i-1} \\
&= -2(1+2i)(-1)^i - (2i-1)(-1)^{i-1} \\
&= (-1)^{i-1}(-2(1+2i)(-1) - (2i-1)) \\
&= (-1)^{i-1}(3+2i) \\
&= (-1)^2(-1)^{i-1}(1+2+2i) \\
&= (1+2(i+1))(-1)^{i+1} \\
&= (1+2n)(-1)^n.
\end{aligned}$$

$P(i+1) = T.$ **[The Inductive Step Holds.]**

Therefore, $\forall n \in D_n, P(n)$ is true. □

Solution 27: Given

$$f_0 = 0, f_1 = 1,$$
$$f_n = f_{n-1} + f_{n-2}, \text{ for } n \geq 2.$$

We want to prove that, for all $n \geq 0$,

$$f_n = \frac{1}{\sqrt{5}} \times \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

- **Inductive Basis:** $n = 0$:

$$\begin{aligned} \tfrac{1}{\sqrt{5}} \times \left(\left(\tfrac{1+\sqrt{5}}{2} \right)^0 - \left(\tfrac{1-\sqrt{5}}{2} \right)^0 \right) &= \tfrac{1}{\sqrt{5}} \times (1-1) \\ &= 0 \ = \ f_0. \end{aligned}$$

$n = 1$:

$$\begin{aligned} \tfrac{1}{\sqrt{5}} \times \left(\left(\tfrac{1+\sqrt{5}}{2} \right)^1 - \left(\tfrac{1-\sqrt{5}}{2} \right)^1 \right) &= \tfrac{1}{\sqrt{5}} \times \left(\tfrac{1+\sqrt{5}}{2} - \tfrac{1-\sqrt{5}}{2} \right) \\ &= \tfrac{1}{\sqrt{5}} \times \left(\tfrac{1+\sqrt{5}-1+\sqrt{5}}{2} \right) \\ &= 1 \ = \ f_1 \end{aligned}$$

[The Basis Holds.]

- **Inductive Hypothesis:** Assume that, for $1 \leq k \leq n$,

$$f_k = \frac{1}{\sqrt{5}} \times \left(\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right).$$

- **Inductive Step:** In the following simplification we combine the two positive terms and the two negative terms, and use

$$\left(\tfrac{1+\sqrt{5}}{2} \right)^2 = \tfrac{3+\sqrt{5}}{2}, \text{ and } \left(\tfrac{1-\sqrt{5}}{2} \right)^2 = \tfrac{3-\sqrt{5}}{2}.$$

$$\begin{aligned} f_{n+1} &= f_n + f_{n-1} \\ &= \tfrac{1}{\sqrt{5}} \left(\left(\tfrac{1+\sqrt{5}}{2} \right)^n - \left(\tfrac{1-\sqrt{5}}{2} \right)^n \right) + \tfrac{1}{\sqrt{5}} \left(\left(\tfrac{1+\sqrt{5}}{2} \right)^{n-1} - \left(\tfrac{1-\sqrt{5}}{2} \right)^{n-1} \right) \\ &= \tfrac{1}{\sqrt{5}} \times \left[\left(\tfrac{1+\sqrt{5}}{2} \right)^n + \left(\tfrac{1+\sqrt{5}}{2} \right)^{n-1} - \left(\tfrac{1-\sqrt{5}}{2} \right)^n - \left(\tfrac{1-\sqrt{5}}{2} \right)^{n-1} \right] \\ &= \tfrac{1}{\sqrt{5}} \times \left[\left(\tfrac{1+\sqrt{5}}{2} \right)^{n-1} \times \left(\tfrac{3+\sqrt{5}}{2} \right) - \left(\tfrac{1-\sqrt{5}}{2} \right)^{n-1} \times \left(\tfrac{3-\sqrt{5}}{2} \right) \right] \\ &= \tfrac{1}{\sqrt{5}} \times \left[\left(\tfrac{1+\sqrt{5}}{2} \right)^{n-1} \times \left(\tfrac{1+\sqrt{5}}{2} \right)^2 - \left(\tfrac{1-\sqrt{5}}{2} \right)^{n-1} \times \left(\tfrac{1-\sqrt{5}}{2} \right)^2 \right] \\ &= \tfrac{1}{\sqrt{5}} \times \left[\left(\tfrac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\tfrac{1-\sqrt{5}}{2} \right)^{n+1} \right] \end{aligned}$$

[The Inductive Step Holds.]

□

Solution 28: Prove by induction that the total number of subsets having exactly two elements in a set of n elements is $n(n-1)/2$.

It is clear that if A is a set of 2 elements, the only subset of A with two elements is A itself. And since $1 = 2(2-1)/2$, the basis holds.

Suppose we have a set A with elements, and let $n \geq 2$. And suppose we already know by the inductive hypothesis that there are $n(n-1)/2$ many subsets of A with 2 elements. Now, we add a new element a into A, and examine the power set of the new A. What are those subsets with 2 elements? They are the old subsets with 2 elements plus every singleton subset of the old A union with $\{a\}$. Apparently, the old A has n-many singleton subsets. Therefore, the new A has $(n(n-1)/2 + n)$-many subsets with 2 elements, where the new size of A is $n+1$. And

$$\frac{n(n-1)}{2} + n = \frac{(n+1)n}{2}.$$

This proves the inductive step and completes the proof of this problem. □

One may also want to define a recurrent relation according to the discussion above, and prove the result by induction more formally. The following is a proof.

Let t(n) be the number of subsets with 2 elements of a set with n elements. Apparently, $t(0) = 0$. If $n \geq 1$, we can recursively define t(n) as follows. For $n \geq 1$,

$$t(n) = t(n-1) + (n-1).$$

Now, let's prove the claim that for all $n \geq 0$, $t(n) = n(n-1)/2$. Define $D_n = \mathbf{N}^0$, and

$$P(n) \quad : \quad t(n) = \frac{1}{2}n(n-1).$$

- **Inductive Basis:** $n = 0$.

$$t(0) = 0 = \frac{1}{2} \times 0 \times (0-1).$$

Therefore, $P(0) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n$.
Suppose, if $n \leq i$, then $P(n) = T$, i.e.

$$t(n) = \frac{1}{2}n(n-1).$$

- **Inductive Step:** Let $n = i + 1$.

$$\begin{aligned} t(n) &= t(i+1) = t(i) + i \\ &= \frac{1}{2}i(i-1) + i = \frac{1}{2}i(i-1+2) \\ &= \frac{1}{2}(i+1)((i+1)-1) = \frac{1}{2}n(n-1). \end{aligned}$$

$P(i + 1) = T$. **[The Inductive Step Holds.]**

Therefore, for any set with n elements, the set has $n(n-1)/2$-many subsets having exactly 2 elements. □

Solution 29: Prove by induction that for all integers $n \geq 2$ the set of all points of intersection of n distinct lines in the plane has no more than $n(n-1)/2$ elements. Give examples showing exactly that many, and also fewer.

Suppose we already have n lines on the plane with the maximum number of intersection points. If we draw a new line on the plane, we cannot introduce more than n new intersection points. Therefore, if $p(n)$ is the maximum number of the intersection points of n distinct lines, the recurrent relation will be

$$p(n) = p(n-1) + (n-1).$$

This is exactly the same as the recurrence relation in the previous problem. We skip the proof here. □

Solution 30: Define $P(n)$ for all $n \geq 1$ as

$$P(n) : 1 \leq x_n \leq \sqrt{n}.$$

- **Inductive Basis:** $n = 1$. $x_1 = 1$, and

$$1 \leq x_1 \leq \sqrt{1}.$$

Therefore, $P(1)$ is true. **[The Basis Holds.]**

- **Inductive Hypothesis:** Assume $P(n)$ is true, i.e.,

$$1 \leq x_n \leq \sqrt{n}.$$

- **Inductive Step:** To prove $P(n + 1)$ is true.

From the hypothesis, we know that $1 \le x_n \le \sqrt{n}$. Thus,

$$\begin{aligned} 1 \le x_n \le \sqrt{n} \Rightarrow\ & 1 \le x_n^2 \le n \\ \Rightarrow\ & 1 + \frac{1}{x_n^2} \le x_n^2 + \frac{1}{x_n^2} \le n + \frac{1}{x_n^2}. \end{aligned}$$

Because $1 \le x_n^2 \Rightarrow \frac{1}{x_n^2} \le 1$, we have

$$1 \le 1 + \frac{1}{x_n^2} \quad \text{and} \quad n + \frac{1}{x_n^2} \le n + 1.$$

Therefore,

$$\begin{aligned} & 1 \le 1 + \frac{1}{x_n^2} \le x_n^2 + \frac{1}{x_n^2} \le n + \frac{1}{x_n^2} \le n + 1 \\ & \Rightarrow 1 \le x_n^2 + \frac{1}{x_n^2} \le n + 1 \\ & \Rightarrow 1 \le x_{n+1}^2 \le n + 1 \\ & \Rightarrow 1 \le x_{n+1} \le \sqrt{n+1}. \end{aligned}$$

$P(n+1)$ is true. **[The Inductive Step Holds.]**

Method 2: There is another way to prove the inductive step.

- **Inductive Step:** To prove $P(n+1)$ is true.

  From the hypothesis we know $1 \le x_n \le \sqrt{n}$, and $\frac{1}{\sqrt{n}} \le \frac{1}{x_n} \le 1$. Moreover, we know $0 \le \frac{1}{\sqrt{n}}$, thus

$$1 \le x_n^2 \le n \tag{3.11}$$

$$\frac{1}{n} \le \frac{1}{x_n^2} \le 1 \tag{3.12}$$

  By adding (11) and (12), we get

$$1 + \frac{1}{n} \le x_n^2 + \frac{1}{x_n^2} \le n + 1.$$

  Because $1 \le 1 + \frac{1}{n}$, we have

$$\begin{aligned} & 1 \le x_n^2 + \frac{1}{x_n^2} \le n + 1 \\ & \Rightarrow 1 \le x_n^2 + \frac{1}{x_n^2} \le n + 1 \\ & \Rightarrow 1 \le x_{n+1}^2 \le n + 1 \\ & \Rightarrow 1 \le x_{n+1} \le \sqrt{n+1}. \end{aligned}$$

  $P(n+1)$ is true. **[The Inductive Step Holds.]**

□

Solution 31:

- **Inductive Basis:** $n = 0$.

$$H_{2^n} = H_1 = \sum_{1\leq i\leq 1} \frac{1}{i} = \frac{1}{1} \geq 1 + \frac{0}{2}.$$

[The Basis Holds.]

- **Inductive Hypothesis:** Assume $H_{2^n} \geq \left(1 + \frac{n}{2}\right)$.
- **Inductive Step:**

$$\begin{aligned}
H_{2^{n+1}} &= H_{2\cdot 2^n} \\
&= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \\
&= H_{2^n} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \\
&\geq 1 + \frac{n}{2} + \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n} \\
&\geq 1 + \frac{n}{2} + \overbrace{\frac{1}{2^n+2^n} + \frac{1}{2^n+2^n} + \cdots + \frac{1}{2^n+2^n}}^{2^n} \\
&= 1 + \frac{n}{2} + \frac{2^n}{2^n+2^n} \\
&= 1 + \frac{n+1}{2}
\end{aligned}$$

[The Inductive Step Holds.]

Therefore, for all $n \geq 0$, $H_{2^n} \geq \left(1 + \frac{n}{2}\right)$. □

Solution 32: Let A^* denote the set of all possible strings made from A, where $\lambda \in A^*$. Let μ range over A^*, and let $|\mu|$ denote the length of μ. We will prove the theorem by mathematical induction *on the length of strings.* Let us first define the domain and the predicate $P(n)$.

$$\begin{aligned}
D_n &: \{0, 1, 2, 3, \ldots\}, \\
P(n) &: \forall \mu \in A^*, [(|\mu| = n) \Rightarrow (\mu \in S \leftrightarrow \mu \text{ is a } \textit{palindrome})].
\end{aligned}$$

- **Inductive Basis:** $n = 0$ and $n = 1$.
  By the definition of S, $\lambda \in S$, and by the definition of palindrome, λ is also a palindrome. Therefore, $P(0) = T$.

  If $|\mu| = 1$, then by rule 2 any single character from A is in P, and it is also a palindrome over A. $P(1) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $i \in D_n$, and $1 \leq i$.
  Suppose if $n \leq i$, then $P(n) = T$, i.e.,

$$\forall \mu \in A^*, [(|\mu| \leq i) \Rightarrow (\mu \in S \leftrightarrow \mu \text{ is a } \textit{palindrome})].$$

- **Inductive Step:** Let $n = i + 1$. Because we assume that $1 \leq i$ in the hypothesis, we have $2 \leq n$. This will simplify our discussion because, as you will see, we don't have to consider rules 1 and 2 in the course of the inductive step. Please note that this is valid, because the cases when $n = 0$ and $n = 1$ were proved in the basis step.

  Let $\mu \in A^*$ and $|\mu| = i + 1$.

  1. If $\mu \in S$, then μ must satisfy rule 3. Rules 1 and 2 are ruled out because $2 \leq |\mu|$. Thus, μ must be a string like $a\nu a$, where $a \in A$ and $\nu \in S$. We also know that $|\nu| = i - 1$, and by the strong inductive hypothesis, $\nu \in S$ if and only if ν is a palindrome. Therefore, ν is a palindrome over A, and $a\nu a$ is also a palindrome over A. Therefore,

  $$\mu \in S \rightarrow \mu \text{ is a } \textit{palindrome}.$$

  2. If μ is a palindrome over A and $2 \leq |\mu|$, μ must be a string like $a\nu a$, where $a \in A$ and ν is a palindrome over A. Since $|\nu| = i - 1$, and by the strong inductive hypothesis, $\nu \in S$ if and only if ν is a palindrome. Therefore, $\nu \in S$, and by rule 3, $a\nu a \in P$. Therefore,

  $$\mu \text{ is a } \textit{palindrome} \rightarrow \mu \in S.$$

  $P(i + 1) = T$. **[The Inductive Step Holds.]**

For any length n, $P(n)$ is true, which means that S is the set of palindromes over A.

Note: The proof given above uses the original form of mathematical induction. We haven't seen the power of structural induction yet. Please compare the following proof with the previous one. We will see the full power of structural induction in the last two problems of this chapter.

Method 2: Structural induction

$$D_s : A^*,$$
$$P(s) : (s \in S \leftrightarrow s \text{ is a } \textit{palindrome}).$$

- **Inductive Basis:** By the definitions of S and palindrome, $\lambda \in S$ and it is a palindrome. Thus, $P(\lambda) = T$. **[The Basis Holds.]**

- **Inductive Hypothesis:** Let $s \in A^*$, and assume $P(s) = T$, i.e.,

$$s \in S \;\leftrightarrow\; s \text{ is a } palindrome.$$

- **Inductive Step:** Our task is to prove

$$asa \in S \;\leftrightarrow\; asa \text{ is a } palindrome.$$

We have two cases about s: (1) $s \in S$, and (2) $s \notin S$.

case 1: $s \in S$. By the definition of S, $asa \in S$. By the hypothesis, s is a palindrome, and hence asa is also a palindrome. Thus,

$$[asa \in S \;\rightarrow\; asa \text{ is a } palindrome] = T.$$

case 2: $s \notin S$. By the definition of S, $asa \notin S$. By the hypothesis, s is not a palindrome, and hence asa is not a palindrome. Thus,

$$[asa \notin S \;\rightarrow\; asa \text{ is not a } palindrome] = T.$$

By contrapositive, we have

$$[asa \text{ is a } palindrome \;\rightarrow\; asa \in S] = T.$$

Together, we have $asa \in S \;\leftrightarrow\; asa$ is a *palindrome*.

[The Inductive Step Holds.]

Therefore, for any string s, $s \in S$ if and only if s is a palindrome. □

Solution 33: Let $D_s = S$, and define two variables predicate $P(\mu, \nu)$ as: for all $\mu, \nu \in S$,

$$P(\mu, \nu) : R(\mu\nu) = R(\nu)R(\mu).$$

- **Inductive Basis:** $\mu = a$, and $\mu = b$.

For $\mu = a$, let $\nu \in S$. We have

$$R(\mu\nu) = R(a\nu) = R(\nu)a = R(\nu)R(a) = R(\nu)R(\mu).$$

Same as $\mu = b$. Thus, for all $\nu \in S$, $P(a, \nu) = T$ and $P(b, \nu) = T$.

[The Basis Holds.]

- **Inductive Hypothesis:** Let $\mu, \nu \in S$. Assume that for all $\xi \in S$, $P(\mu, \xi) = P(\nu, \xi) = T$, i.e.,

$$R(\mu\xi) = R(\xi)R(\mu) \ \ \& \ \ R(\nu\xi) = R(\xi)R(\nu).$$

- **Inductive Step:** Our task is to prove that for all $\xi \in S$, $P(\mu\nu, \xi)$ is true, i.e., $R(\mu\nu\xi) = R(\xi)R(\mu\nu)$. Let $\xi \in S$,

$$\begin{aligned} R(\mu\nu\xi) &= R(\mu(\nu\xi)) & \\ &= R(\nu\xi)R(\mu) & [\textbf{by IH}] \\ &= R(\xi)R(\nu)R(\mu) & [\textbf{by IH}] \\ &= R(\xi)(R(\nu)R(\mu)) & \\ &= R(\xi)R(\mu\nu) & [\textbf{by IH}] \end{aligned}$$

Therefore, for all $\mu, \nu \in S$, $R(\mu\nu) = R(\nu)R(\mu)$. □

Solution 34: Let $D_s = S$, and define predicate $P(\mu)$ as: for all $\mu \in S$,

$$P(\mu) : R(R(\mu)) = \mu.$$

We will use the result proved in the previous problem: for all $\mu, \nu \in S$,

$$R(\mu\nu) = R(\nu)R(\mu). \tag{3.13}$$

- **Inductive Basis:** $\mu = a$, and $\mu = b$. It is clear that $P(a) = P(b) = T$, because

$$R(R(a)) = R(a) = a, \text{ and } R(R(b)) = R(b) = b.$$

[The Basis Holds.]

- **Inductive Hypothesis:** Let $\mu, \nu \in S$. Assume that

$$R(R(\mu)) = \mu, \text{ and } R(R(\nu)) = \nu.$$

- **Inductive Step:** We want to prove that $R(R(\mu\nu)) = \mu\nu$.

$$\begin{aligned} R(R(\mu\nu)) &= R(R(\nu)R(\mu)) & \text{by (13)} \\ &= R(R(\mu))R(R(\nu)) & \text{by (13)} \\ &= \mu\nu & [\textbf{by IH}] \end{aligned}$$

[The Inductive Step Holds.]

Therefore, for all $\mu \in S$, $R(R(\mu)) = \mu$. □

Solution 35:

We will prove this problem by structural mathematical induction. For convenience, let $\alpha(\omega)$ denote the number of a's in ω, and $\beta(\omega)$ the number of b's in ω. We shall prove that, for every $\omega \in A$, $\alpha(\omega) = \beta(\omega)$.

- **Inductive Basis:** $\omega = \Lambda$. It is clear that $\alpha(\omega) = \beta(\omega) = 0$. **[The Basis Holds.]**
- **Inductive Hypothesis:** Let $\mu, \nu \in A$, and $\alpha(\mu) = \beta(\mu) = m$, and $\alpha(\omega) = \beta(\omega) = n$.
- **Inductive Step:**

  By rule 2: $\omega = a\mu b$. Clearly, $\alpha(\omega) = \beta(\omega) = m + 1$.

  By rule 3: $\omega = \mu\nu$. Thus, $\alpha(\omega) = \alpha(\mu\nu) = m + n$ and $\beta(\omega) = \beta(\mu\nu) = m + n$.

  Thus, $\alpha(\omega) = \beta(\omega)$ in all cases. **[The Inductive Step Holds.]**

Therefore, if $\omega \in A$, then $\alpha(\omega) = \beta(\omega)$. □

Solution 36: As with the previous problem, let $\alpha(\omega)$ denote the number of a's in ω, and $\beta(\omega)$ the number of b's in ω. We shall prove that, if $\omega \in A$, then for any prefix σ of ω we have $\alpha(\sigma) \geq \beta(\sigma)$.

- **Inductive Basis:** $\omega = \Lambda$. The only prefix of Λ is Λ itself. It is clear that $\alpha(\omega) = \beta(\omega) = 0$. Thus, the inductive basis holds. **[The Basis Holds.]**
- **Inductive Hypothesis:** Fix $\mu, \nu \in A$, and assume that, for any σ, if σ is a prefix of μ or ν, then $\alpha(\sigma) \geq \beta(\sigma)$.
- **Inductive Step:**

  By rule 2: We obtain a new string $\omega = a\mu b$. Given σ a prefix of ω, we have the following cases.

  1. $\sigma = \Lambda$. In this case, it is clear that $\alpha(\sigma) \geq \beta(\sigma)$.
  2. $\sigma = a\sigma'$, where σ' is a prefix of μ. Thus, $\alpha(\sigma) = \alpha(a\sigma') = \alpha(\sigma') + 1$ and $\beta(\sigma) = \beta(a\sigma') = \beta(\sigma')$. By the inductive hypothesis, $\alpha(\sigma') \geq \beta(\sigma')$. Thus, $\alpha(\sigma') + 1 \geq \beta(\sigma')$, and hence $\alpha(\sigma) \geq \beta(\sigma)$.
  3. $\sigma = \omega = a\mu b$. In this case, $\alpha(\sigma) = \alpha(\mu) + 1$ and $\beta(\sigma) = \beta(\mu) + 1$. By the inductive hypothesis, $\alpha(\mu) \geq \beta(\mu)$. Thus, $\alpha(\sigma) \geq \beta(\sigma)$.

By rule 3: We obtain a new string $\omega = \mu\nu$. Given σ a prefix of ω, we have the following cases.

1. σ is a prefix of μ. By the inductive hypothesis, $\alpha(\sigma) \geq \beta(\sigma)$.
2. $\sigma = \mu\xi$, where ξ is a prefix of ν. By the inductive hypothesis, we have $\alpha(\mu) \geq \beta(\mu)$ and $\alpha(\xi) \geq \beta(\xi)$. Thus, $\alpha(\mu) + \alpha(\xi) \geq \beta(\mu) + \beta(\xi)$. Since $\alpha(\sigma) = \alpha(\mu) + \alpha(\xi)$ and $\beta(\sigma) = \beta(\mu) + \beta(\xi)$, it follow that $\alpha(\sigma) \geq \beta(\sigma)$.

[**The Inductive Step Holds.**]

This completes the proof. □